Probabilistic Algorithms: Exercise 4

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Definition 1 (permutations and determinant). *Let π be a permutation on $[n]$. The permutation π^{-1} is defined as follows: On input i, $\pi^{-1}(i) = j$ where j is such that $\pi(j) = i$. A pair $i, j \in [n]$ is an inversion of π if $i < j$ and $\pi(i) > \pi(j)$. The sign of π denoted by $sign(\pi)$ is the parity of the number of inversions of π. It is easy to check that for any π_1, π_2, $sign(\pi_1 \circ \pi_2) = sign(\pi_1) \cdot sign(\pi_2)$ where $\pi_1 \circ \pi_2(i) = \pi_1(\pi_2(i))$.*

The determinant of an $n \times n$ matrix A is given by:

$$\sum_{\pi} sign(\pi) \prod_{1 \le i \le n} A_{i\pi(i)}$$

In class we saw an algorithm based on polynomial identity testing that decides whether a *bipartite* graph has a perfect matching. The purpose of this exercise is to generalize the algorithm to any graph.

Let n be an even number and let $G = (V, E)$ be an (undirected) graph on n vertices. A perfect matching in such a graph is a permutation π such that $\pi = \pi^{-1}$ and for every $i \in [n]$, $(i, \pi(i)) \in E$.

For $1 \le i, j \le n$ let X_{ij} be a variable. We define the following $n \times n$ matrix A where entries are either numbers or variables. The entry A_{ij} is given by:

- 0 if $(i, j) \notin E$.
- X_{ij} if $(i, j) \in E$ and $i < j$.
- $-X_{ji}$ if $(i, j) \in E$ and $i > j$.

Let $p_G(X_{11}, \ldots, X_{nn})$ be the determinant of A. Observe that p_G is a polynomial of degree at most n in the variables $X_{11}, \ldots, X_{nn}$.

1. (Finding a matching in a (non-bipartite) graph)

   (a) Show that if G has a perfect matching then $p_G \not\equiv 0$.

   (b) A permutation π contains a cycle of length k if there are $i_1, \ldots, i_k$ for such that for every $\ell < k$, $\pi(i_\ell) = i_{\ell+1}$ and furthermore $\pi(i_k) = i_1$. Show that the contribution of permutations π that contain a cycle of odd length to p_G is zero. (Hint: For every permutation π that contains a cycle of odd length let π^R denote the (unique) permutation in which the cycle is "reversed". Show that the terms of the two permutations cancel.)

   (c) Let σ be a permutation on $[n]$ that is a cycle. such that for every i, $(i, \sigma(i)) \in E$ show that G has a perfect matching. (Hint: use the fact that n is even).

   (d) Let π be a permutation that does not contain an odd cycle, and that for every i, $(i, \pi(i)) \in E$. Show that G has a perfect matching. (Hint: generalize the answer to the previous question.)

(e) Give a probabilistic polynomial time Monte-Carlo algorithm for testing whether G has a perfect matching.

(f) Give a probabilistic polynomial time algorithm such that if G contains a perfect matching then the algorithm outputs a perfect matching π with probability at least $1/2$ and if G does not contain a perfect matching then the algorithm outputs "no matching" with probability one.