Probabilistic Algorithms: Exercise 2

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1. (MAX-CUT) Given a graph $G = (V, E)$, a cut C is a subset of V such that C is not empty and $C \neq V$. The size of the cut denoted by $Size(C)$ is the number of edges that go from vertices in C to vertices outside of C. (It is NP-hard to find the maximal size cut).

   (a) Give a probabilistic algorithm that on a given graph G outputs a cut C such that $E(Size(c)) \geq |E|/2$.

   (b) Use the method of conditional probabilities to get a deterministic algorithm that on a given graph G outputs a cut C with $Size(C) \geq |E|/2$.

2. (Routing on the cube by random reordering) The cube is the graph whose vertices are $\{0,1\}^n$ and two vertices are connected if they agree in exactly one position. In class we've seen Valiant's Algorithm for routing on the cube that with high probability routes all packets in time $O(n)$. In this question we consider a different probabilistic algorithm:

   Algorithm: RR (random reordering)
   Given a permutation σ over $\{0,1\}^n$ we need to define a path from every node $i \in \{0,1\}^n$ to node $j = \sigma(i)$. Let S be the set of indices on which i and j differ. (The bit-changing algorithm we've seen in class uses a path of length $|S|$ where at step ℓ it flips the ℓ'th bit in S.[1]) In the new algorithm each node chooses a random permutation $\pi : \{1 \ldots |S|\} \to S$ and at step ℓ flips the $\pi(\ell)$'th bit.

   Show that there is a permutation σ that causes this algorithm to take time $2^{\Omega(n)}$ with probability at Least $3/4$. (Hint: Consider the permutation $\sigma(a \circ b) = (b \circ a)$ that we've seen in class. Consider the nodes of the form $a \circ 0^{n/2}$ for all a with exactly $k = n/8e$ ones. Show that the expected number of nodes sending their packets through $0^{n/2} \circ 0^{n/2}$ is high. You may want to use the inequality:

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

3. (finding an independent set in a graph with few edges) Given a graph $G = (V, E)$ an independent set is a subset $S \subseteq V$ such that for every two vertices $v_1, v_2 \in S$, v_1 and v_2 are not connected. (It is NP-hard to find the maximal independent set).

   (a) Give an algorithm that for any graph G finds an independent set of size $|V| - |E|$.

   (b) Let G be a graph with n vertices and $nd/2$ edges (where $d > 1$ is some number). Consider the following algorithm. Go over all vertices in the graph and for each vertex, delete it and all it's edges with probability $1 - 1/d$.
   Compute the expected number of remaining vertices and edges after this process.

   (c) Give a probabilistic algorithm that given a graph G with n vertices and $nd/2$ edges outputs an independent set such that the expected size of this set is at least $n/2d$.

   (d) Use the method of conditional probabilities to get a deterministic algorithm that given such a graph G outputs an independent set of size $n/2d$.

[1] Recall that we've seen that the bit-changing algorithm may take exponential time.