Complexity Theory : Exercise 1

Submit until 22/3

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Please write clear and precise answers. A 10 point bonus is given for printed solutions.

1. (Reminder of NP and NP completeness). Consider the languages:

$$Clique = \{(G,k) : \text{G is a graph that has a clique of size } k\}$$

$$Half-Clique = \{G : \text{G is a graph that has a clique of size } |V|/2\}$$

   (a) Show that $Half-Clique \in NP$.
   (b) Show that $Clique \leq_L Half-Clique$.

2. (Composition of logspace functions).

   (a) Let M_1, M_2 be machines that run in space $O(\log n)$ and compute functions f_1 and f_2 respectively. Consider the function $f(x) = f_2(f_1(x))$. Show that this function is computable by a machine that runs in space $O(\log n)$.
   (b) Show that if $L_1 \leq_L L_2$ and $L_2 \leq_L L_3$ then $L_1 \leq_L L_3$.
   (c) Show that if $L_1 \leq_L L_2$ and $L_2 \in L$ then $L_1 \in L$.
   (d) Show that if $L_1 \leq_L L_2$ and $L_2 \in NL$ then $L_1 \in NL$.

3. Show that $TQBF \in PSPACE$. (Recall that $TQBF$ is the set of all true quantified boolean formulae).

4. ($2-SAT \in NL$) Consider the language $2-SAT = \{\phi : \phi \text{ is a satisfiable 2-CNF formula}\}$. (Reminder: a 2-CNF formula is a conjunction of clauses where each clause is the disjunction of two literals).

   Given a 2-CNF formula ϕ on n variables $X_1, \ldots, X_n$ we associate with it a directed graph G_ϕ where the nodes are all the $2n$ literals and for every clause $(\ell_i \vee \ell_j)$ (where ℓ_i and ℓ_j are literals) we add the edges $\neg\ell_i \to \ell_j$ and $\neg\ell_j \to \ell_i$.

   (a) Show that ϕ is not satisfiable if and only if there is a literal ℓ such that there are paths from ℓ to $\neg\ell$ and from $\neg\ell$ to ℓ.
   (b) Show that $2-SAT \in NL$.