Foundation of Cryptography, Lecture 1
One-Way Functions

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(Slightly edited by Ronen Shaltiel, all errors are by Ronen Shaltiel)

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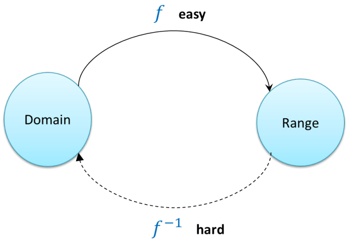
2018

Section 1

One-Way Functions

Informal discussion



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- Easy to compute, everywhere
- Hard to invert, on the average

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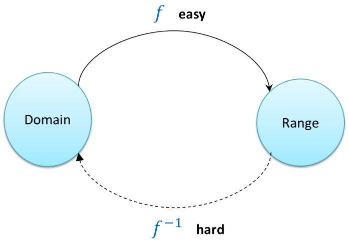

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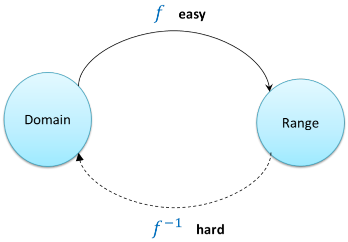

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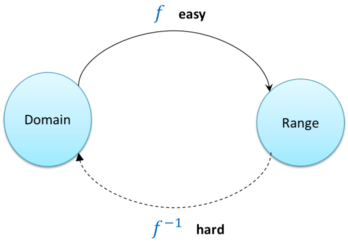

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- Why should we care about OWFs?
- Hidden in (almost) any cryptographic primitive: necessary for “cryptography"
- Sufficient for many cryptographic primitives

“Application”: Authentication where server doesn’t store the user’s password.

Formal definition

Definition 1 (one-way functions (OWFs))

A polynomial-time computable function $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

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We typically omit 1^n from the input list of A

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7. Non uniform OWFs

Definition 2 (Non-uniform OWF))

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is non-uniformly one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[C_n(f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

for any polynomial-size family of circuits $\{C_n\}_{n \in \mathbb{N}}$.

Length-preserving functions

Definition 3 (length preserving functions)

A function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0,1\}^*$

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Assume that OWFs exit, then there exist length-preserving OWFs.

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Theorem 4

Assume that OWFs exit, then there exist length-preserving OWFs.

Proof idea: use the assumed OWF to create a length preserving one.

Partial domain functions

Definition 5 (Partial domain functions)

Let $m, \ell \colon \mathbb{N} \mapsto \mathbb{N}$ be polynomials. Let $f \colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{m(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n \in \mathbb{N}}$, and maps strings of length $\ell(n)$ to strings of length $m(n)$.

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The definition of one-wayness naturally extends to such (efficient) functions.

OWFs imply length-preserving OWFs cont.

Let $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in \mathrm{poly}$ be a bound on its computing-time, and assume wlg. that p is monotony increasing (can we?). Note that $|f(x)| \leq p(|x|)$.

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Construction 6 (the length preserving function)

Define $g\colon \{0,1\}^{p(n)+1} \mapsto \{0,1\}^{p(n)+1}$ as

$$g(x) = f(x_{1,\ldots,n}), 1, 0^{p(n)-|f(x_{1,\ldots,n})|}$$

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Answer: using reduction.

Proving that g is one-way

Proof: Assume that g is not one-way. Namely, there exists PPT A, $q \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \{p(n)+1 \colon n \in \mathbb{N}\}$, with

$$\Pr_{x \leftarrow \{0,1\}^{n'}} \left[\mathsf{A}(1^{n'}, y) \in g^{-1}(g(x)) \right] > 1/q(n') \tag{1}$$

for every $n' \in \mathcal{I}$.

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We show how to use A for inverting f.

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it follows that $f(w_{1,\ldots,n}) = y$ (?).□

Algorithm 9 (Inverter B for f)

Input: 1^n and $y \in \{0,1\}^*$

1. Let $x = A(1^{p(n)+1}, y, 1, 0^{p(n)-|y|})$
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1. $\mathcal{I}'$ is infinite
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&= \Pr_{x' \leftarrow \{0,1\}^{p(n)+1}}\left[\mathsf{A}(1^{p(n)+1}, g(x'))_{1,\ldots,n} \in f^{-1}(f(x'_{1,\ldots,n}))\right]
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Algorithm 9 (Inverter B for f)

Input: 1^n and $y \in \{0,1\}^*$

1. Let $x = \mathsf{A}(1^{p(n)+1}, y, 1, 0^{p(n)-|y|})$
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Claim 10

Let $\mathcal{I}' := \{n \in \mathbb{N} \colon p(n) + 1 \in \mathcal{I}\}$. Then

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2. $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{B}(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n)+1)$ for every $n \in \mathcal{I}'$

This contradicts the assumed one-wayness of f. □

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From partial-domain length-preserving OWFs to length-preserving OWFs

Construction 11

Given a function $f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{\ell(n)}$, define $f_{\mathrm{all}}\colon \{0,1\}^n \mapsto \{0,1\}^n$ as

$$f_{\mathrm{all}}(x) = f(x_{1,\ldots,k}), 0^{n-k}$$

where $n = |x|$ and $k := \max\{\ell(n') \leq n \colon n' \in [n]\}$.

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We conclude that the existence of OWF implies the existence of length-preserving OWF that is defined over all input lengths.

Few remarks

More “security-preserving" reductions exits.

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Convention for rest of the talk

Let $f \colon \{0,1\}^n \mapsto \{0,1\}^n$ be a one-way function.

Weak one-way functions

Definition 13 (weak one-way functions)

A poly-time computable function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is α-one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] \leq \alpha(n)$$

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3. Can we “amplify" weak OWF to strong ones?

Strong to weak OWFs

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Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way

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Proof: For a OWF f, let

$$g(x,b) = \begin{cases} (1, f(x)), & b = 1; \\ (0, x), & \text{otherwise } (b = 0). \end{cases}$$

Weak to strong OWFs

Theorem 15 (weak to strong OWFs (Yao))

Assume there exist $(1-\delta)$*-weak OWFs with* $\delta(n) \geq 1/q(n)$ *for some* $q \in \mathrm{poly}$*, then there exist (strong) one-way functions.*

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  Computing Ax takes $\Theta(n^2)$ times, but computing $A(x_1, x_2, \ldots, x_n)$ takes …

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- Fortunately, parallel repetition does amplify weak OWFs :-)

Amplification via parallel repetition

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Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ *be a* $(1-\delta)$*-weak OWF for* $\delta(n) = 1/q(n)$ *for some (positive)* $q \in \mathrm{poly}$*, and let* $t(n) = \left\lceil \frac{\log^2 n}{\delta(n)} \right\rceil$*. Then* $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ *defined by* $g(x_1,\ldots,x_{t(n)}) = f(x_1),\ldots,f(x_{t(n)})$*, is a one-way function.*

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Proving that g is One-Way – the Naive approach

Assume A attacks each of the t outputs of g independently: $\exists$ PPT A′ such that $\mathsf{A}(z_1, \ldots, z_t) = \mathsf{A}'(z_1) \ldots, \mathsf{A}'(z_t)$

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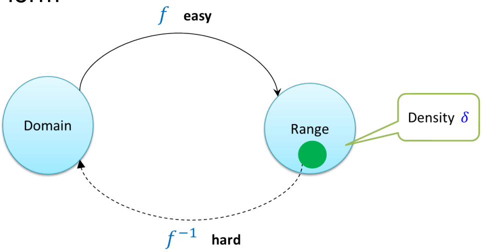
Any idea?

Hardcore sets

Assume f is of the form



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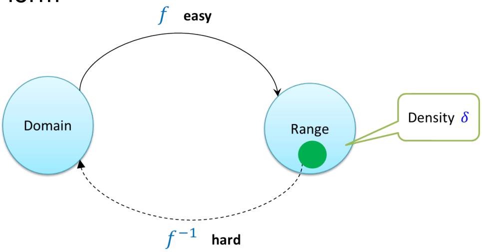

Definition 17 (hardcore sets)

$\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$ is a δ-hardcore set for $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, if:

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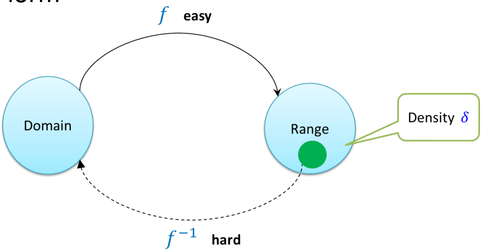
Hardcore sets

Assume f is of the form



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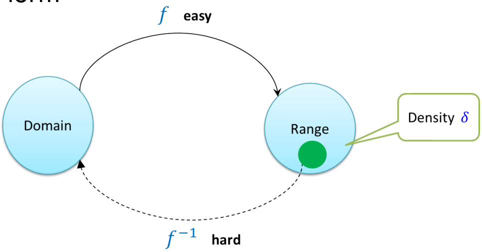

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Unfortunately, we do not know how to prove that f has hardcore set :-<

Failing sets

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Definition 18 (failing sets)

$f\colon \{0,1\}^n \mapsto \{0,1\}^n$ has a δ-failing set for a pair (A, q) of algorithm and polynomial, if exists $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$, such that the following holds for large enough n:

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1. If this set is small, show that A inverts f very well.
2. If this set is large, then it is by definition a fooling set.

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Proof: Assume $\exists$ PPT A and $q \in \mathrm{poly}$, such that for any $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$ at least one of the following holds:

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We'll use A to contradict the hardness of f.

Using A to invert f

For $n \in \mathbb{N}$, let $\mathcal{S}_n := \{y \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] < 1/q(n)\}$.

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The second item cannot hold, therefore the first item must hold, meaning that:

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Namely, f is not $(1-\delta)$-one-way $\square$

g is not one-way $\implies$ f has no $\delta/2$ failing set

We show: g is not one way $\implies$ f has no $\delta/2$ failing-set for some PPT B and $q \in \text{poly}$.

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Assume $\exists$ PPT A, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n)\cdot n}} \left[\mathsf{A}(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}$$

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  $\overset{\text{(for large enough } n)}{\geq} \frac{1}{2t(n)p(n)}$
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Namely, f has no $\delta/2$ failing set for $(\mathsf{B}, q = 2t(n)p(n))$

The no failing-set algorithm: Proof of main claim

Algorithm 24 (Inverter B on input $y \in \{0,1\}^n$)

1. Choose $w \leftarrow (\{0,1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
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&= \mathop{\mathsf{E}}_r \left[\alpha_r(n) \right] / t(n) - n^{-\log n} \\
&\geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}.
\end{aligned}
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- What properties of the weak OWFs have we used in the proof?