Foundation of Cryptography, Lecture 3
Hardcore Predicates for Any One-way Function

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Can we find a function of x that is totally unpredictable — looks uniform — given $f(x)$?

Such functions have many cryptographic applications

Formal definition

Definition 1 (hardcore predicates)

A poly-time computable $b\colon \{0,1\}^n \mapsto \{0,1\}$ is an hardcore predicate of $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, if

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{P}(f(x)) = b(x)] \leq \frac{1}{2} + \mathsf{neg}(n)$$

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  Answer to above is positive, in case f is one-to-one

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$$g(x,i) = f(x), i$$

Assuming f *is one way, then*

$$\Pr_{x \leftarrow \{0,1\}^n, i \leftarrow [n]}[\mathsf{A}(f(x),i) = x_i] \leq 1 - 1/2n$$

for any PPT A*.*

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The resulting predicate is not for f but for (the one-way function) g^t ...

The Goldreich-Levin Hardcore predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r \rangle_2 := (\sum_{i=1}^n x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^n x_i \cdot r_i$.

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- Note that if f is one-to-one, then so is g.

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Proof by reduction: a PPT A for predicting $b(x,r)$ "too well" from $(f(x), r)$, implies an inverter for f

Section 1

Proving GL – The information theoretic case

Min entropy

Definition 4 (min-entropy)

The min entropy of a random variable (or distribution) X, is defined as

$$\mathsf{H}_\infty(X) := \min_{y \in \mathsf{Supp}(X)} \log \frac{1}{\Pr_X[y]}.$$

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Examples:

- Z is uniform over a set of size 2^k.
- $Z = X|_{f(X)=y}$, where $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is 2^k to 1 , $y \in f(\{0,1\}^n) := \{f(x)\colon x \in \{0,1\}^n\}$ and X is uniform over $\{0,1\}^n$.

  Equivalently, $X \leftarrow f^{-1}(y)$.

In both cases, $H_\infty(Z) = k$.

Pairwise independent hashing

Definition 5 (pairwise independent function family)

A function family $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ is pairwise independent, if $\forall$ $x \neq x' \in \{0,1\}^n$ and $y, y' \in \{0,1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}}[h(x) = y \land h(x') = y')] = 2^{-2m}$.

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Lemma 6 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_\infty(X) \geq k$ and let $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be pairwise independent, then

$$\mathrm{SD}((H, H(X)), (H, U_m)) \leq 2^{(m-k-2))/2},$$

where H is uniformly distributed over $\mathcal{H}$ and U_m is uniformly distributed over $\{0,1\}^m$.

See proof here, page 13.

Efficient function families

Definition 7 (efficient function families)

An ensemble of function families $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is efficient, if

Samplable. Exists PPT that given 1^n, outputs (the description of) a uniform element in $\mathcal{F}_n$.

Efficient. Exists poly-time algorithm that given $x \in \{0,1\}^n$ and (a description of) $f \in \mathcal{F}_n$, outputs $f(x)$.

Proving GL for compressing functions

Definition 8

Function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is $d(n)$ regular, if $\left|f^{-1}(y)\right| = d(n)$ for every $y \in f(\{0,1\}^n)$.

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Lemma 9

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a $d(n) \in 2^{\omega(\log n)}$ regular function, and let $\mathcal{H} = \{\mathcal{H}_n\}$ be an efficient family of Boolean pairwise independent functions over $\{0,1\}^n$. Define $g\colon \{0,1\}^n \times \mathcal{H}_n \mapsto \{0,1\}^n \times \mathcal{H}_n$ as

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How does it relate to Goldreich-Levin?
$\{\mathcal{H}_n = \{b_r(\cdot) = b(r,\cdot)\}_{r\in\{0,1\}^n}\}$ is (almost) pairwise independent.

Proving Lemma 9

The lemma follows by the next claim (?)

Claim 10

$\mathrm{SD}\left((f(U_n), H, H(U_n)), (f(U_n), H, U_1)\right) = \mathrm{neg}(n)$, where $H = H_n$ is uniformly distributed over $\mathcal{H}_n$.

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Proving Lemma 9, cont.

Since $\mathsf{H}_\infty(X_y) = \log(d(n))$ for any $y \in f(\{0,1\}^n)$,

Proving Lemma 9, cont.

Since $\mathsf{H}_\infty(X_y) = \log(d(n))$ for any $y \in f(\{0,1\}^n)$, the leftover hash lemma (Lemma 6) yields that

$$\begin{aligned}\mathsf{SD}((H, H(X_y)), (H, U_1)) &\leq 2^{(1-\mathsf{H}_\infty(X_y)-2))/2} \\ &= 2^{(1-\log(d(n)))/2} = \mathsf{neg}(n). \quad \square\end{aligned}$$

Section 2

Proving GL – The Computational Case

Proving Goldreich-Levin Theorem

Theorem 11 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$*, define* $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ *as* $g(x,r) = (f(x),r)$*.*

If f *is one-way, then* $b(x,r) := \langle x,r\rangle_2$ *is an hardcore predicate of* g*.*

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If f is one-way, then $b(x,r) := \langle x,r\rangle_2$ is an hardcore predicate of g.

Proof: Assume $\exists$ PPT A, $p \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\Pr[\mathsf{A}(g(U_n,R_n)) = b(U_n,R_n)] \geq \frac{1}{2} + \frac{1}{p(n)}, \tag{1}$$

for any $n \in \mathcal{I}$, where U_n and R_n are uniformly (and independently) distributed over $\{0,1\}^n$.

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for any $n \in \mathcal{I}$, where U_n and R_n are uniformly (and independently) distributed over $\{0,1\}^n$.

We show $\exists$ PPT B and $q \in \text{poly}$ with

$$\Pr_{y \leftarrow f(U_n)}[\mathsf{B}(y) \in f^{-1}(y)] \geq \frac{1}{q(n)}, \tag{2}$$

for every $n \in \mathcal{I}$.

Proving Goldreich-Levin Theorem

Theorem 11 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ as $g(x,r) = (f(x), r)$.

If f is one-way, then $b(x,r) := \langle x, r\rangle_2$ is an hardcore predicate of g.

Proof: Assume $\exists$ PPT A, $p \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\Pr[\mathsf{A}(g(U_n, R_n)) = b(U_n, R_n)] \geq \frac{1}{2} + \frac{1}{p(n)}, \tag{1}$$

for any $n \in \mathcal{I}$, where U_n and R_n are uniformly (and independently) distributed over $\{0,1\}^n$.

We show $\exists$ PPT B and $q \in \text{poly}$ with

$$\Pr_{y \leftarrow f(U_n)}[\mathsf{B}(y) \in f^{-1}(y)] \geq \frac{1}{q(n)}, \tag{2}$$

for every $n \in \mathcal{I}$. In the following fix $n \in \mathcal{I}$.

Focusing on a good set

Claim 12

There exists a set $\mathcal{S} \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}|}{2^n} \geq \frac{1}{2p(n)}$, and
2. $\Pr[\mathsf{A}(f(x), R_n) = b(x, R_n)]] \geq \frac{1}{2} + \frac{1}{2p(n)}, \forall x \in S$.

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Proof: Let $\mathcal{S} := \{x \in \{0,1\}^n \colon \Pr[\mathsf{A}(f(x), R_n) = b(x, R_n)] \geq \frac{1}{2} + \frac{1}{2p(n)}\}$.

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Proof: Let $\mathcal{S} := \{x \in \{0,1\}^n \colon \Pr[A(f(x), R_n) = b(x, R_n)] \geq \frac{1}{2} + \frac{1}{2p(n)}\}$.

$$\Pr[A(g(U_n, R_n)) = b(U_n, R_n)] \leq \Pr[U_n \notin \mathcal{S}] \cdot \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}]$$

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$$\begin{aligned}\Pr[\mathsf{A}(g(U_n, R_n)) = b(U_n, R_n)] &\leq \Pr[U_n \notin \mathcal{S}] \cdot \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}] \\ &\leq \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}] \Box\end{aligned}$$

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We conclude the theorem's proof showing exist $q \in \mathrm{poly}$ and PPT B:

$$\Pr[\mathsf{B}(f(x)) \in f^{-1}(f(x)) \geq \frac{1}{q(n)}, \tag{3}$$

for every $x \in \mathcal{S}$.

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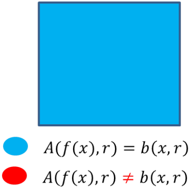
We conclude the theorem's proof showing exist $q \in \text{poly}$ and PPT B:

$$\Pr[\mathsf{B}(f(x)) \in f^{-1}(f(x)) \geq \frac{1}{q(n)}, \qquad (3)$$

for every $x \in \mathcal{S}$. In the following we fix $x \in \mathcal{S}$.

The Perfect Case

$\Pr[A(f(x), R_n) = b(x, R_n)] = 1$



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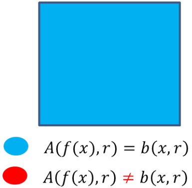

In particular, $\mathsf{A}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, where $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

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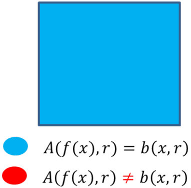

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Hence, $x_i = \langle x, e^i \rangle_2$

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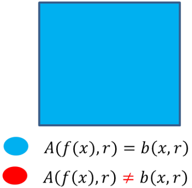

In particular, $\mathsf{A}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, where $e^i = (\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{n-i})$.

Hence, $x_i = \langle x, e^i \rangle_2 = b(x, e^i) = \mathsf{A}(f(x), e^i)$

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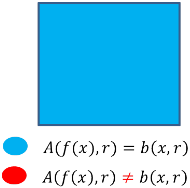


In particular, $\mathsf{A}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, where $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

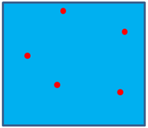
Hence, $x_i = \langle x, e^i \rangle_2 = b(x, e^i) = \mathsf{A}(f(x), e^i)$

Algorithm 13 (Inverter B on input y)

Return $(\mathsf{A}(y, e^1), \ldots, \mathsf{A}(y, e^n))$.

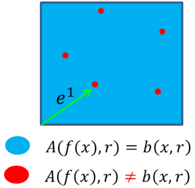
Easy case

$$\Pr[A(f(x), R_n) = b(x, R_n)] \geq 1 - \text{neg}(n)$$



- $A(f(x), r) = b(x, r)$ (blue)
- $A(f(x), r) \neq b(x, r)$ (red)

Easy case



$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] \geq 1 - \mathrm{neg}(n)$$

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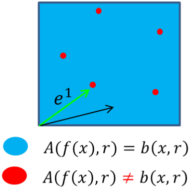

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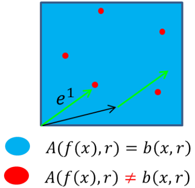

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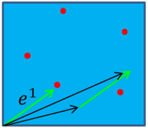

$A(f(x), r) = b(x, r)$

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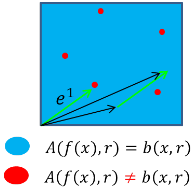

$$\Pr[A(f(x), R_n) = b(x, R_n)] \geq 1 - \text{neg}(n)$$

Fact 14

1. $b(x,w) \oplus b(x,y) = b(x, w \oplus y)$ *for every* $w, w, y \in \{0,1\}^n$.

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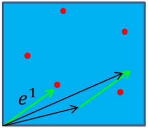

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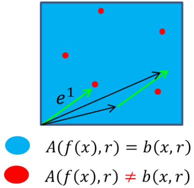

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1. *$b(x,w) \oplus b(x,y) = b(x, w \oplus y)$ for every $w, w, y \in \{0,1\}^n$.*
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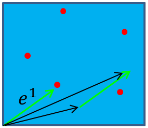
Hence, $\forall i \in [n]$:

1. $x_i = b(x, e^i) = b(x, r) \oplus b(x, r \oplus e^i)$ for every $r \in \{0,1\}^n$

Easy case



$$\Pr[A(f(x), R_n) = b(x, R_n)] \geq 1 - \text{neg}(n)$$

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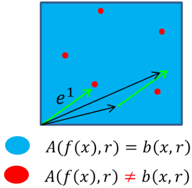
Hence, $\forall i \in [n]$:

1. $x_i = b(x, e^i) = b(x, r) \oplus b(x, r \oplus e^i)$ for every $r \in \{0,1\}^n$
2. $\Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \geq 1 - \text{neg}(n)$

Easy case



$$\Pr\left[\mathsf{A}(f(x), R_n) = b(x, R_n)\right] \geq 1 - \mathrm{neg}(n)$$

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Hence, $\forall i \in [n]$:

1. $x_i = b(x, e^i) = b(x, r) \oplus b(x, r \oplus e^i)$ for every $r \in \{0,1\}^n$
2. $\Pr[\mathsf{A}(f(x), R_n) = b(x, R_n) \wedge \mathsf{A}(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \geq 1 - \mathrm{neg}(n)$

Algorithm 15 (Inverter B on input y)

Return $(\mathsf{A}(y, R_n) \oplus \mathsf{A}(y, R_n \oplus e^1)), \ldots, \mathsf{A}(y, R_n) \oplus \mathsf{A}(y, R_n \oplus e^n))$.

Proving Fact 14

1. For $w, w, y \in \{0,1\}^n$:

$$\begin{aligned} b(x,y) \oplus b(x,w) &= \left(\bigoplus_{i=1^n} x_i \cdot y_i\right) \oplus \left(\bigoplus_{i=1^n} x_i \cdot w_i\right) \\ &= \bigoplus_{i=1^n} x_i \cdot (y_i \oplus w_i) \\ &= b(x, y \oplus w) \end{aligned}$$

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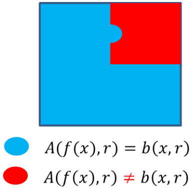
&= \bigoplus_{i=1^n} x_i \cdot (y_i \oplus w_i) \\
&= b(x, y \oplus w)
\end{aligned}
$$

2. For $r, y \in \{0,1\}^n$:

$$
\Pr\left[R_n \oplus r = y\right] = \Pr\left[R_n = y \oplus r\right] = 2^{-n}
$$

Intermediate Case

$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] \geq \frac{3}{4} + \frac{1}{q(n)}$$



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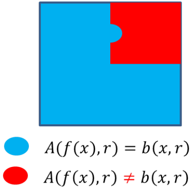

For any $i \in [n]$

$$\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i]$$
$$\geq \quad \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)]$$

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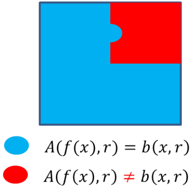

For any $i \in [n]$

$$\begin{aligned} &\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \\ \geq \quad & \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \\ \geq \quad & 1 - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) \end{aligned}$$

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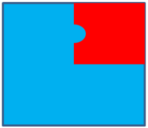
$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] \geq \frac{3}{4} + \frac{1}{q(n)}$$



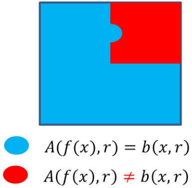
For any $i \in [n]$

$$\begin{aligned} & \Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \\ \geq \quad & \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \\ \geq \quad & 1 - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) = \frac{1}{2} + \frac{2}{q(n)} \end{aligned}$$

- (blue): $A(f(x), r) = b(x, r)$
- (red): $A(f(x), r) \neq b(x, r)$

Intermediate Case

$$\Pr\left[A(f(x),R_n)=b(x,R_n)\right] \geq \frac{3}{4}+\frac{1}{q(n)}$$



For any $i \in [n]$

$$\begin{aligned}
&\Pr[A(f(x),R_n)\oplus A(f(x),R_n\oplus e^i)=x_i]\\
\geq\quad & \Pr[A(f(x),R_n)=b(x,R_n)\wedge A(f(x),R_n\oplus e^i)=b(x,R_n\oplus e^i)]\\
\geq\quad & 1-\left(1-(\frac{3}{4}+\frac{1}{q(n)})\right)-\left(1-(\frac{3}{4}+\frac{1}{q(n)})\right)=\frac{1}{2}+\frac{2}{q(n)}
\end{aligned}$$

Algorithm 16 (Inverter B on input $y \in \{0,1\}^n$)

1. For every $i \in [n]$
   - 1.1 Sample $r^1,\ldots,r^v \in \{0,1\}^n$ uniformly at random
   - 1.2 Let $m_i = \mathrm{maj}_{j\in[v]}\{(A(y,r^j)\oplus A(y,r^j\oplus e^i)\}$
2. Output $(m_1,\ldots,m_n)$

B's Success Provability

The following claim holds for “large enough" $v = v(n) \in \text{poly}(n)$.

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Claim 17

For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \text{neg}(n)$.

B's Success Provability

The following claim holds for "large enough" $v = v(n) \in \mathrm{poly}(n)$.

Claim 17

For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \mathrm{neg}(n)$.

Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff
$A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.

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Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff $A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.
We want to lowerbound $\Pr\left[\sum_{j=1}^{v} W^j > \frac{v}{2}\right]$.

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For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \text{neg}(n)$.

Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff $A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.
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We complete the proof taking $X^j = W^j$, $\varepsilon = 1/4q(n)$ and $v \in \omega(\log(n) \cdot q(n)^2)$.

The actual (hard) case

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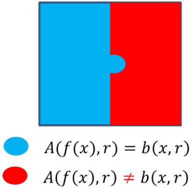

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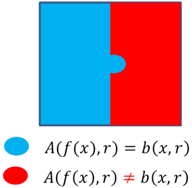

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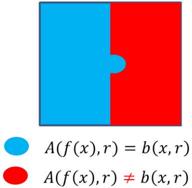

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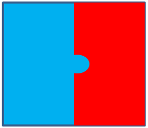

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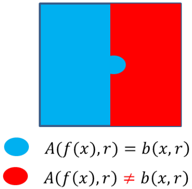

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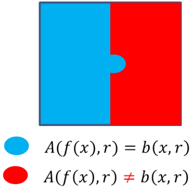

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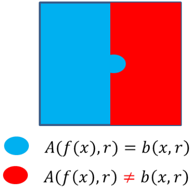

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  Solution: choose the samples in a correlated manner

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1. Sample uniformly (and independently) $t^1, \ldots, t^\ell \in \{0,1\}^n$
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- Problem: the $W^\mathcal{L}$'s are dependent!

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1. $\forall \mathcal{L} \subseteq [\ell]$, $R^{\mathcal{L}}$ is uniformly distributed over $\{0,1\}^n$.
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Proof: (1) is clear, we prove (2) in the next slide.

Proving Fact 20(2)

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□

Pairwise independence variables

Definition 21 (pairwise independent random variables)

A sequence of random variables $X^1, \ldots, X^v$ is pairwise independent, if $\forall i \neq j \in [v]$ and $\forall a, b$, it holds that

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Lemma 22 (Chebyshev's inequality)

Let $X^1, \ldots, X^v$ be pairwise-independent random variables with expectation μ and variance σ^2. Then, for every $\varepsilon > 0$,

$$\Pr\left[\left|\frac{\sum_{j=1}^{v} X^j}{v} - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{\varepsilon^2 v}$$

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Taking the guessing into account, yields that B outputs x with probability at least $2^{-\ell}/2 \in \Omega(n/q(n)^2)$.

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$\implies$ (by GL) Exists algorithm B that guesses X from nothing, with prob $\alpha^{O(1)} > 2^{-t}$

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  The difference comparing to Goldreich-Levin – no control over the R_n's.