UNSUPERVISED LEARNING 2011

LECTURE :ICA

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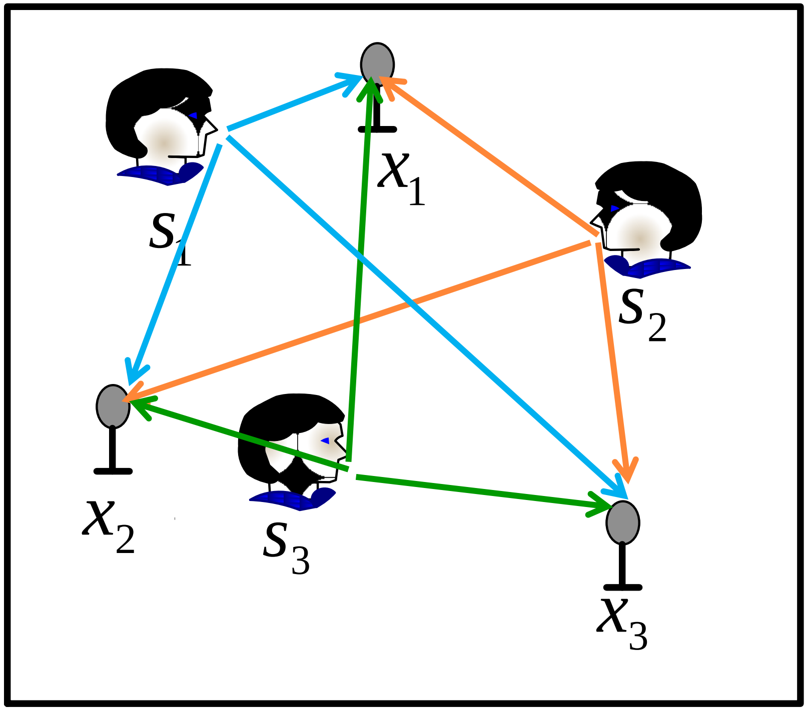
Based on Lecture Notes by A. Ng

Cocktail Party



- microphone signals are mixed speech signals

$$x_1(t) = a_{11}s_1(t) + a_{12}s_2(t) + a_{13}s_3(t)$$
$$x_2(t) = a_{21}s_1(t) + a_{22}s_2(t) + a_{23}s_3(t)$$
$$x_3(t) = a_{31}s_1(t) + a_{32}s_2(t) + a_{33}s_3(t)$$

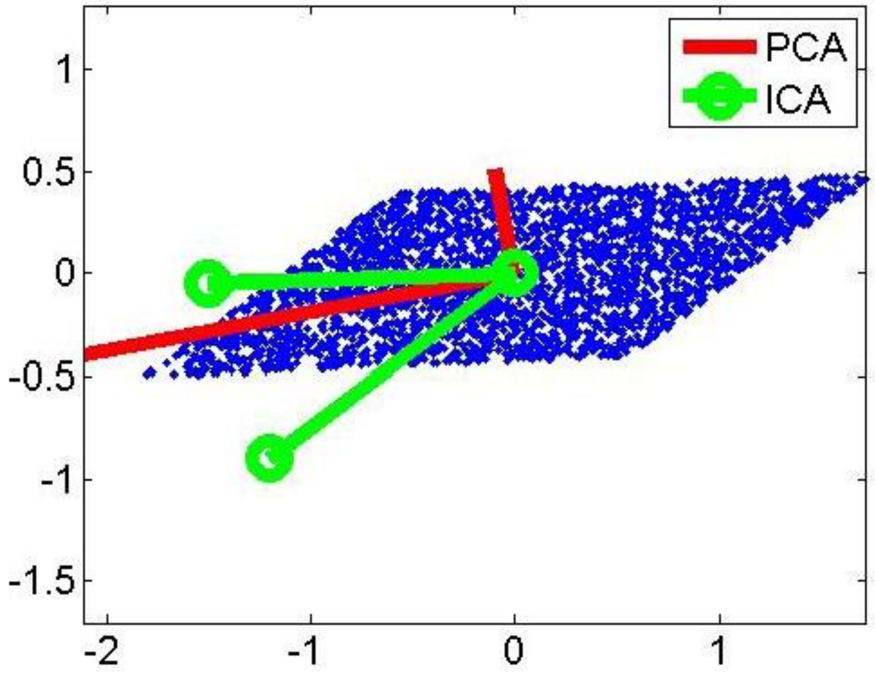
- Input: microphone signals x_1, x_2, x_3
- Goal: recover the speech signals s_1, s_2, s_3

http://research.ics.tkk.fi/ica/cocktail/cocktail_en.cgi

ICA vs. PCA

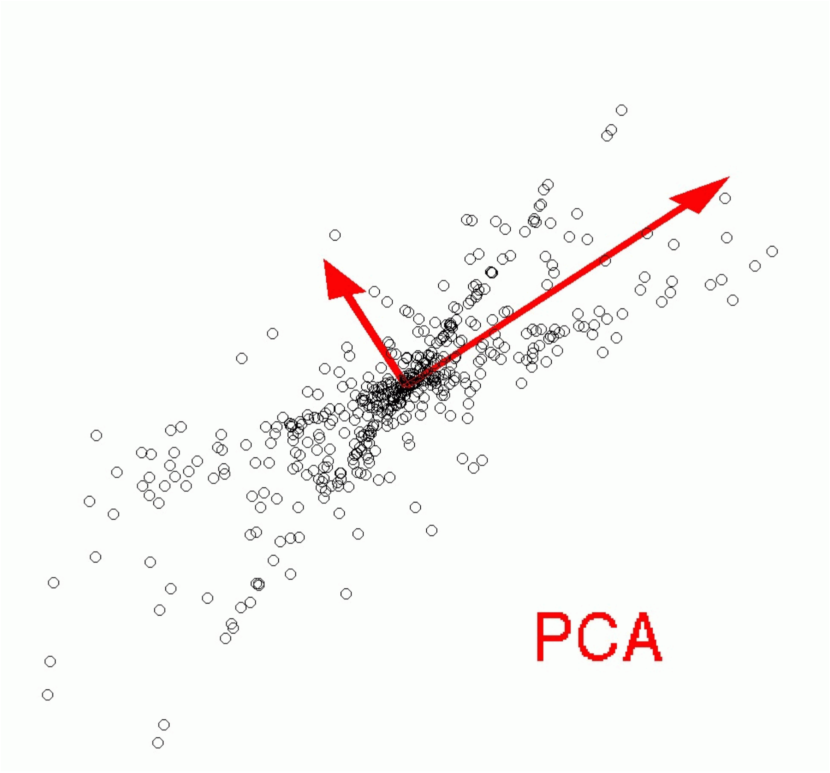
- Similar to PCA
  - Finds a new basis to represent the data
- Different from PCA
  - PCA removes only correlations**,** ICA removes correlations, **and higher order dependence.**
  - In PCA some components are **more important than others** (based on eigenvalues) in ICA components are **equally important.**

ICA vs. PCA

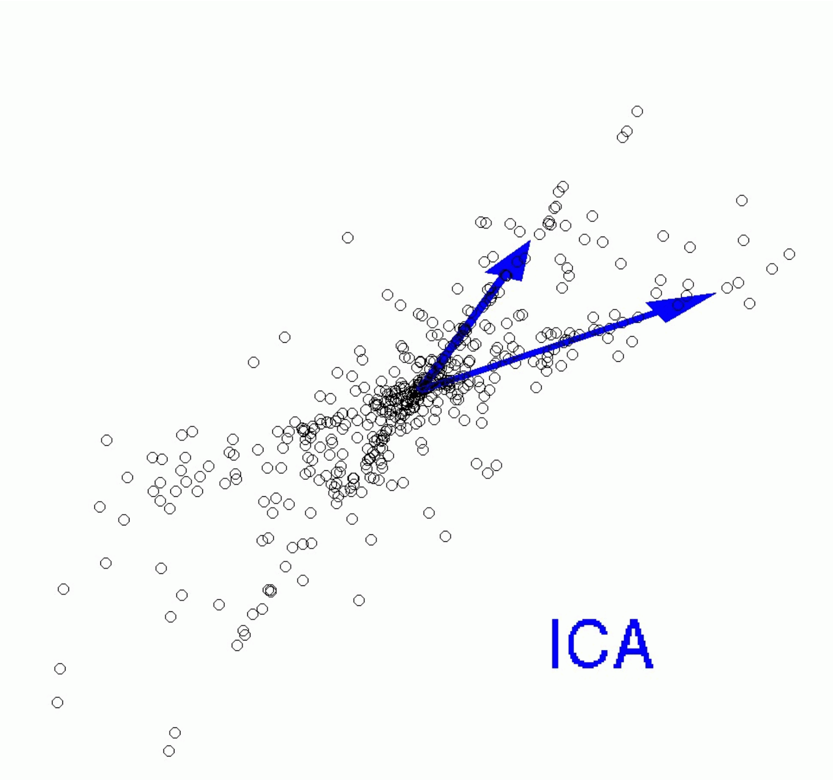


- PCA: principle components are orthogonal.
- ICA: independent components are not!

ICA vs. PCA



maximal variance directions



independent components

Model

- Assume data $s \in R^n$, generated by n independent sources.

- We assume:

$$x = As,$$

↓

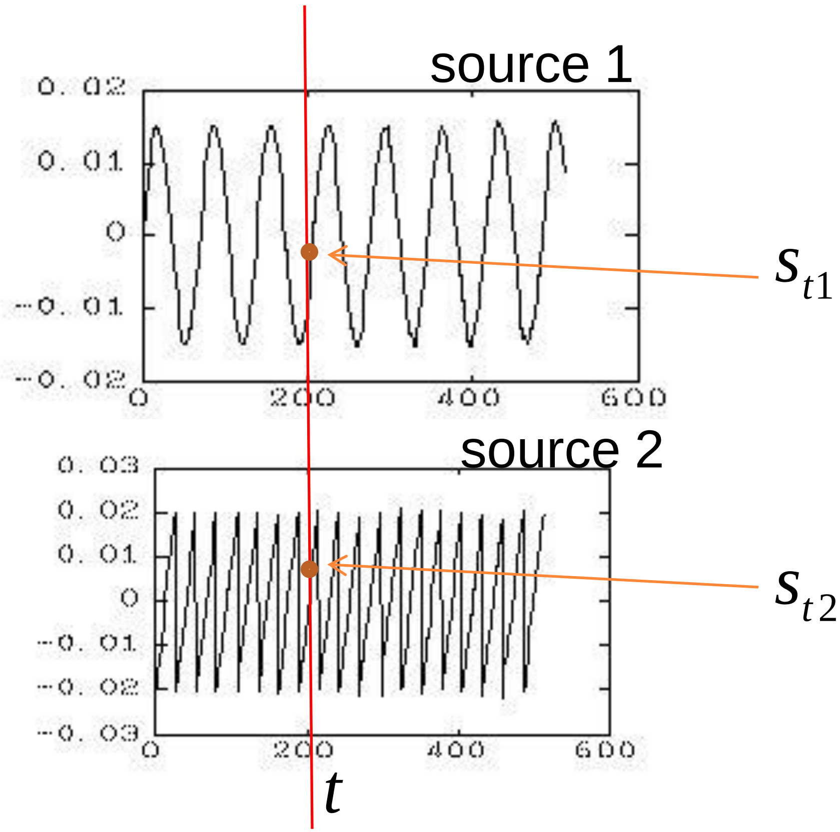
mixing matrix

$A \in R^{n \times n}$ is unknown

Model

- Assume data $s \in R^n$, generated by n independent sources.

  s_{ij} signal from source j at time i.



mic. j at time i

$$x_{ij} = \sum_{k=1}^{n} A_{jk} s_{ik}$$

sum over sources

Problem Definition

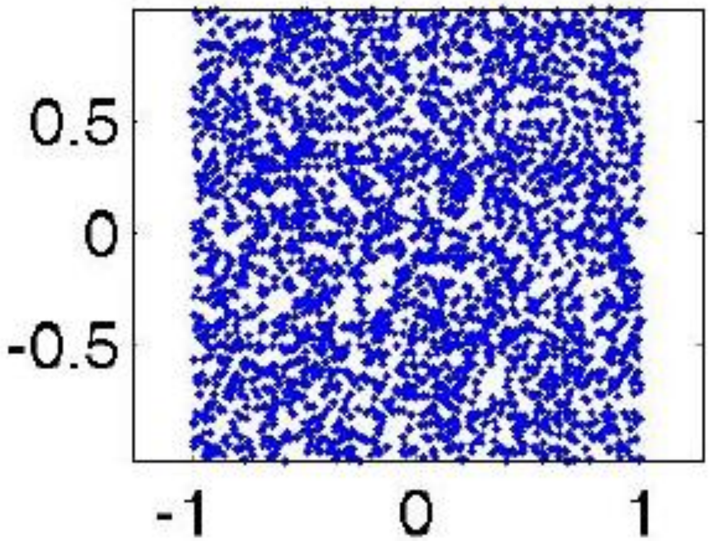
- We observe $\{x_i;\ i = 1,..,m\}$ *i* denotes time
- Goal: recover the sources s_j, that generated the data $(x = As)$.

- Let $W = A^{-1}$ unmixing matrix
- Goal is to find W, such that $s_i = Wx_i$
- Denote

$$W = \begin{bmatrix} - w_1^T - \\ \vdots \\ - w_n^T - \end{bmatrix}$$

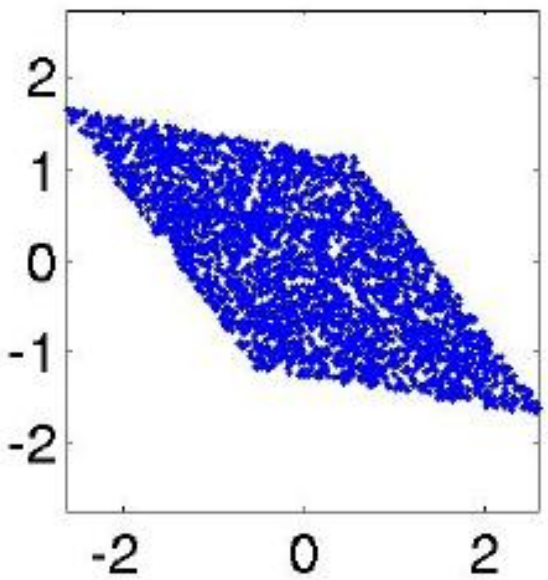
then the *j*-th source can be recovered by $s_{ij} = w_j^T x_i$

ICA Intuition



original

$s_j \in \text{Uniform}[-1,1]$



mixed

ICA Ambiguities

- If we have no prior knowledge about the mixing matrix, then there are inherent ambiguities in A that are impossible to recover.
- The sources can be recovered up to
  - Permutation
  - Scaling
  - Sign

Permutation Ambiguity

Assume that P is a nxn permutation matrix.

Examples: $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}; \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad PW = \begin{bmatrix} w_{21} & w_{22} \\ w_{11} & w_{12} \end{bmatrix}$$

Given only the x_i's , we cannot distinguish between W and PW.

The permutation of the original sources is ambiguous.

Not important in most applications

Scaling Ambiguity

$$x_i = As_i$$

$$A \to 2, \quad s_i \to (0.5 s_i) \quad \Rightarrow \quad x_i = 2A(0.5 s_i)$$

$$A \to \begin{bmatrix} | & & | & \\ a_1 & \cdots & \alpha a_j & \cdots \\ | & & | & \end{bmatrix}, \; s_j \to 1/\alpha \, s_j \quad \Rightarrow \quad x_i = \begin{bmatrix} | & & | & \\ a_1 & \cdots & \alpha a_j & \cdots \\ | & & | & \end{bmatrix} \begin{bmatrix} s_{i1} \\ \vdots \\ 1/\alpha s_{ij} \\ \vdots \end{bmatrix}$$

We cannot recover the "correct" scaling of the sources.

Not important in most applications

Scaling a speaker's speech signal s_j by some positive factor affects only the volume of that speaker's speech.

Also, sign changes do not matter: s_j and $-s_j$ sound identical when played on a speaker.

Gaussian sources are problematic

$$n = 2, s \sim N(0, I), \quad x = As$$

⇩

$$x \sim N(0, AA^T)$$

$$E[xx^T] = E[Ass^T A^T] = AA^T$$

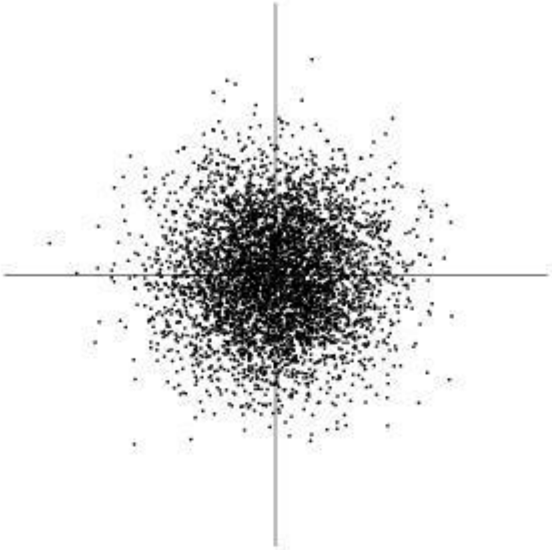


Figure 7: The multivariate distribution of two independent gaussian variables.

Let R be an arbitrary orthogonal matrix, such that $RR^T = R^T R = I$.

Let $A' = AR$, then $x' = A's$ ⇨ $x' \sim N(0, AA^T)$

$$E[x'x'^T] = E[A'ss^T A'^T] = E[ARss^T (AR)^T] = ARR^T A = AA^T$$

Gaussian Sources are Problematic

- Whether the mixing matrix is A or A', we would observe data from a $N(0, AA^T)$ distribution.
- Thus, there is no way to tell if the sources were mixed using A or A'.
- There is an arbitrary rotational component in the mixing matrix that cannot be determined from the data, and we cannot recover the original sources.
- Reason: The Gaussian distribution is **spherically symmetric.**
- For non-Gaussian data, it is possible, given enough data, to recover the n independent sources.

Densities and linear transformations

Suppose s is a r.v drawn according to $p_s(s)$.

Let $x \in R$ be a r.v. defined by $x = As$. The density of x is given by:

$$p_x(x) = p_s(Wx) \cdot |W|$$

where $W = A^{-1}$ (A is squared invertible matrix)

Example: $s \sim \text{Uniform}[0,1] : p_s(s) = 1 \, (0 \le s \le 1)$

Let $A = 2$, then $x = 2s$. Clearly, $x \sim \text{Uniform}[0,2]$

Thus, $p_x(x) = 0.5 \, (0 \le x \le 2)$.

ICA algorithm

- Assume that the distribution of s_i is $p_s(s_i)$.
- The joint distribution is

$$p(s) = \prod_{j=1}^{n} p_s(s_j)$$

sources are independent

- Using the previous formulation, we can derive

$$p(x) = \prod_{j=1}^{n} p_s(w_j^T x)|W|$$

$x = As = W^{-1}s$

$p(x) = p_s(Wx) \cdot |W|$

- We must specify a density for the individual sources p_s.

ICA algorithm

- A cumulative distribution of a real r.v. z is defined by

$$F(z_0) = P\left(z \le z_0\right) = \int_{-\infty}^{z_0} p_z(z)dz$$

- The density of z can be found by $p_z(z) = F'(z)$.

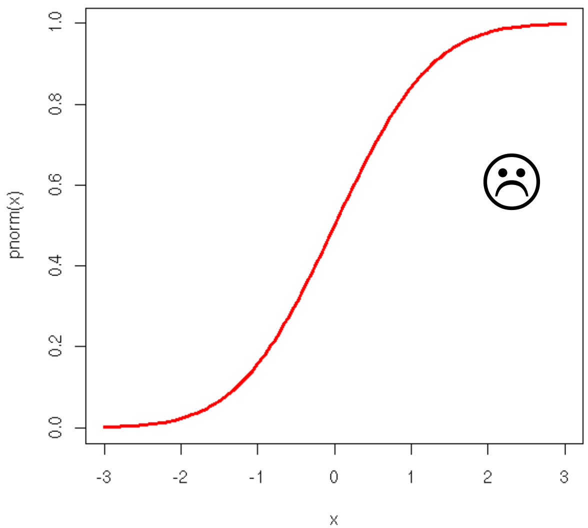
Specify a density for the s_j ⇨ specify its cdf.

If you have a prior knowledge that the sources' densities take a certain form, then use it here, otherwise make an assumption about cdf.

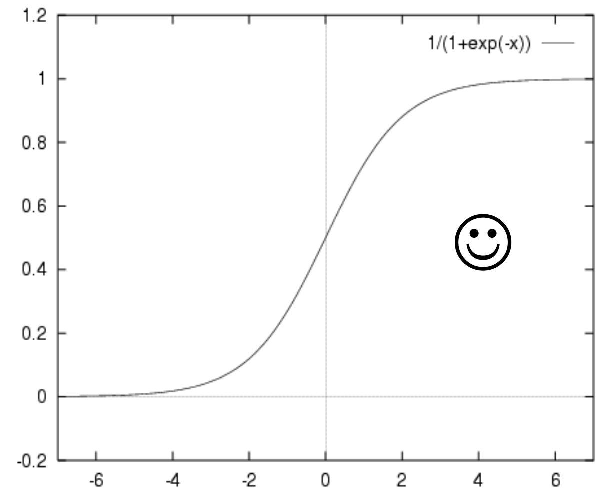
Density of s

cdf is has to be a monotonic function that increases from zero to one.

Gaussian CDF



sigmoid



$$g(s) = 1/(1+e^{-s})$$

$$p(s) = g'(s)$$

We assume that the data x_i has zero mean. This is necessary because our assumption that $p(s) = g'(s)$ implies $E(s) = 0$. Thus $E(x) = E(As) = 0$

ICA algorithm

- W is a parameter of our model that we want to estimate.
- Given a training set $\{x_i; i = 1,\ldots, m\}$, the log likelihood is:

$$l(W) = \sum_{i=1}^{m}\left(\sum_{j=1}^{s}\log g'\left(w_j^T x_i\right) + \log|W|\right).$$

- Maximize $l(W)$ using gradient ascent:

$W \leftarrow W + \eta \nabla l(W),$ where η is the learning rate.

Equivalently, $w_j \leftarrow w_j + \eta \boxed{\frac{\partial}{\partial w_j} l(W)}$?

ICA algorithm

- By taking the derivatives of $l(W)$ using:

  $$g(x) = 1/(1+e^{-x});\ \ g'(x) = g(x)(1-g(x))$$

  $$\nabla_W |W| = |W| \left(W^{-1}\right)^T$$

  we obtain the update rule:

  $$W \leftarrow W + \eta \left(\begin{bmatrix} 1-2g\left(w_1^T x_i\right) \\ 1-2g\left(w_2^T x_i\right) \\ \vdots \\ 1-2g\left(w_n^T x_i\right) \end{bmatrix} x_i^T + \left(W^T\right)^{-1} \right)$$

- When the algorithm converges, compute $s_i = W x_i$.

Remarks

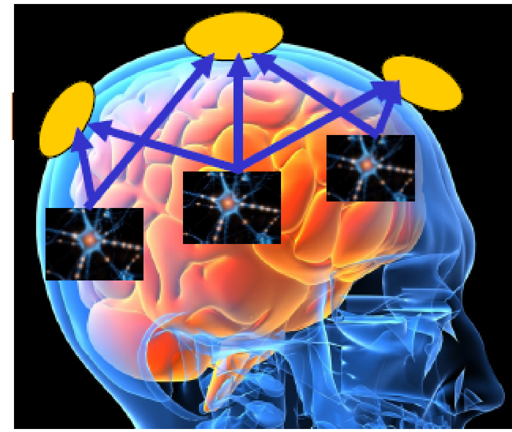
- We assumed that $\{x_i; i = 1,\ldots, m\}$ are independent of each other.
- This assumption is incorrect for time series where the x_i's are dependent (e.g. speech data).
- it can be shown, that having correlated training examples will not hurt the performance of the algorithm if we have sufficient data.
- Tip: run stochastic gradient ascent on a randomly shuffled copy of the training set.

Application domains of ICA

- Blind source separation
- Image denoising
- Medical signal processing – fMRI, ECG, EEG
- Modelling of the hippocampus and visual cortex
- Feature extraction, face recognition
- Compression, redundancy reduction
- Watermarking
- Clustering
- Time series analysis (stock market, microarray data)
- Topic extraction
- Econometrics: Finding hidden factors in financial data

Slide due B. Poczos

ICA Application,
Removing Artifacts from EEG



- EEG ~ *Neural cocktail party*
- Severe ***contamination*** of EEG activity
  - eye movements
  - blinks
  - muscle
  - heart, ECG artifact
  - vessel pulse
  - electrode noise
  - line noise, alternating current (60 Hz)
- ICA can improve signal
  - effectively ***detect, separate and remove*** activity in EEG records from a wide variety of artifactual sources.
    (Jung, Makeig, Bell, and Sejnowski)
- ICA weights help find **location** of sources

Slide due B. Poczos

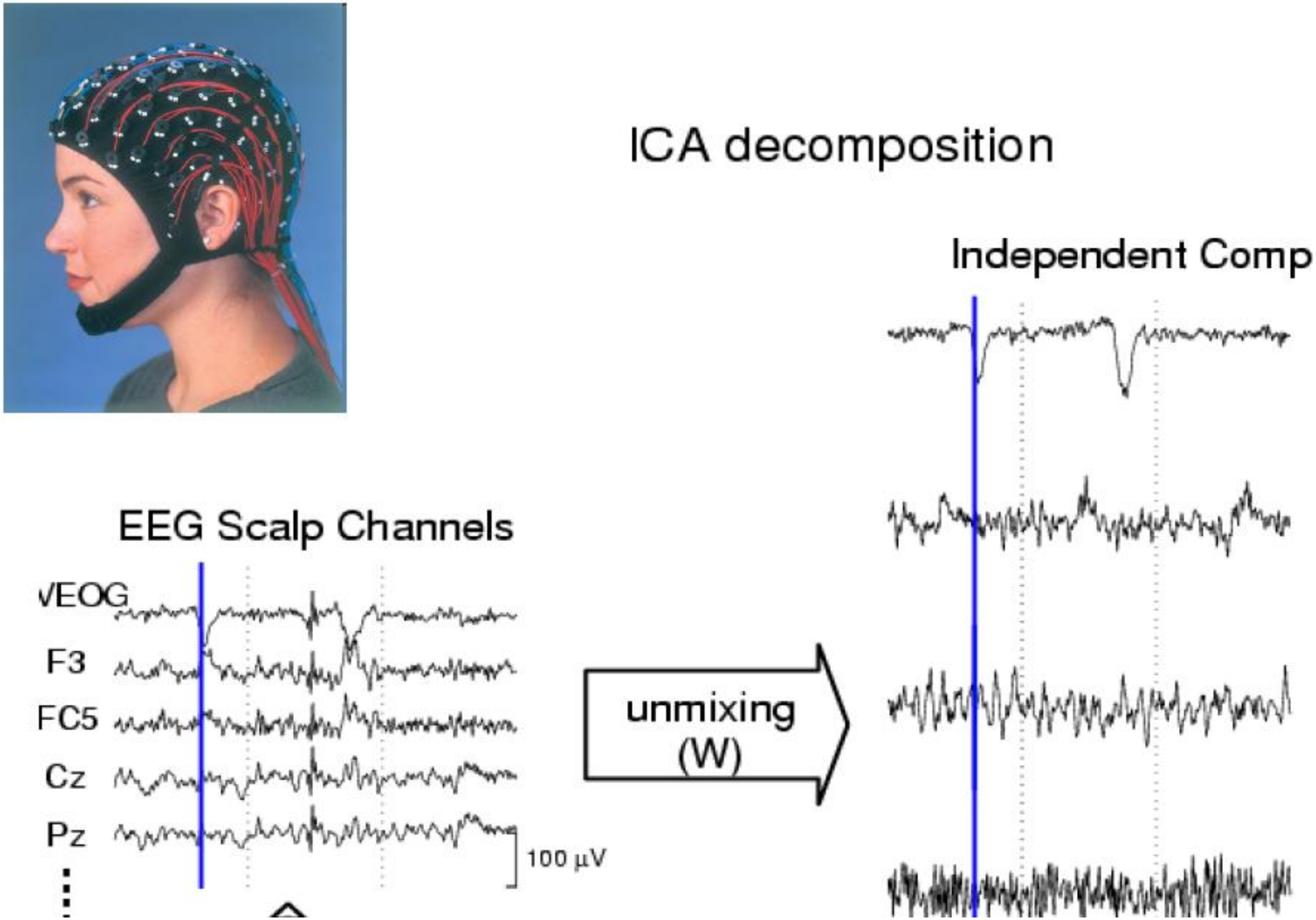


Fig. from Jung

Summed Projection of Selected Components

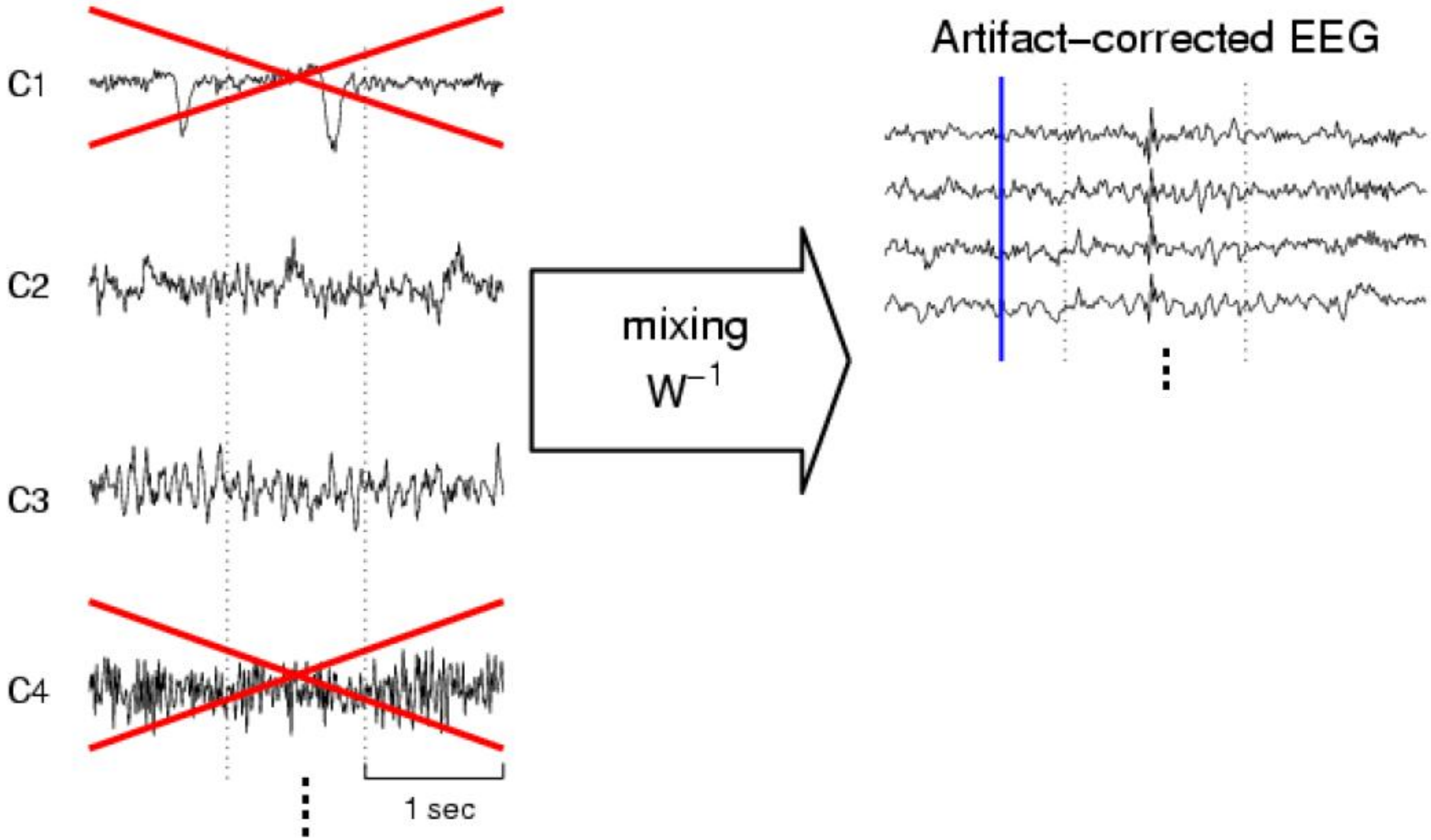


Fig from Jung

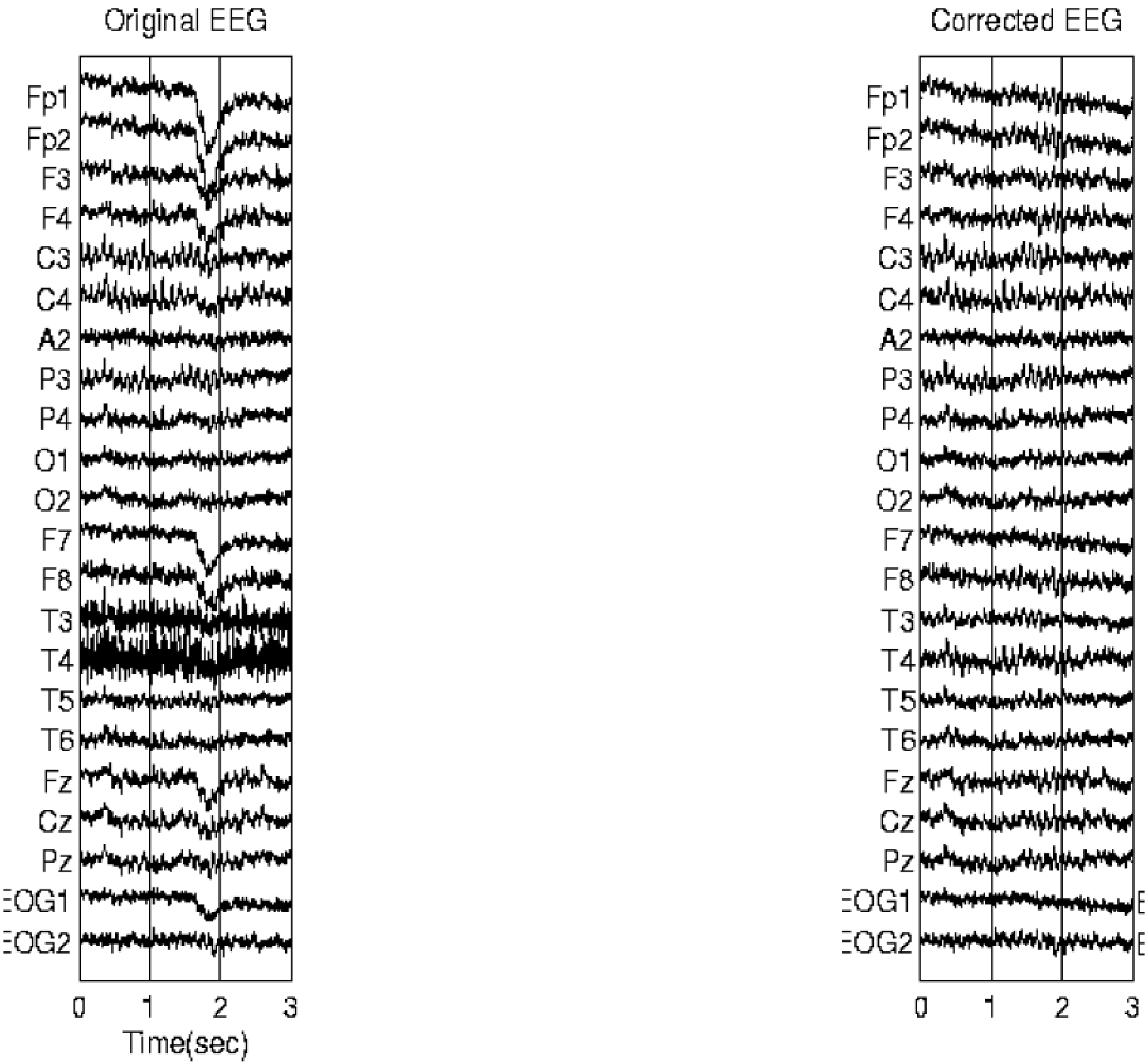
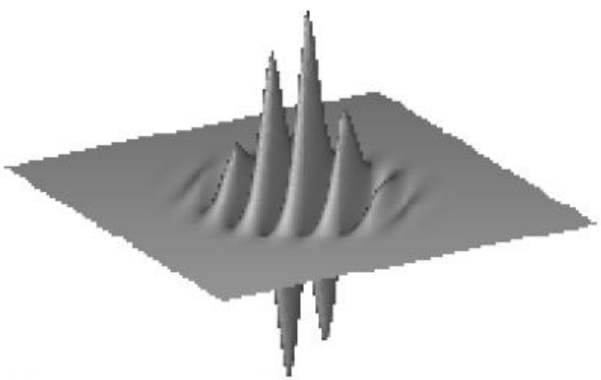
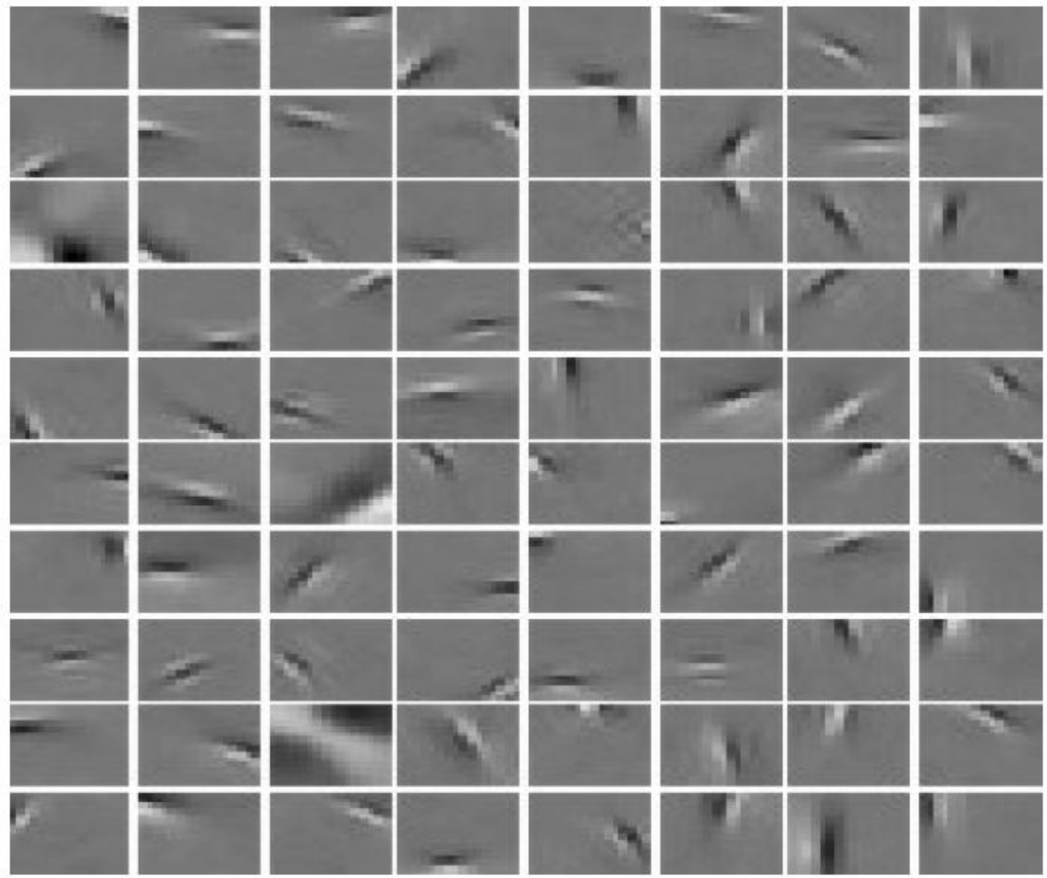


Fig from Jung

ICA basis vectors extracted from natural images



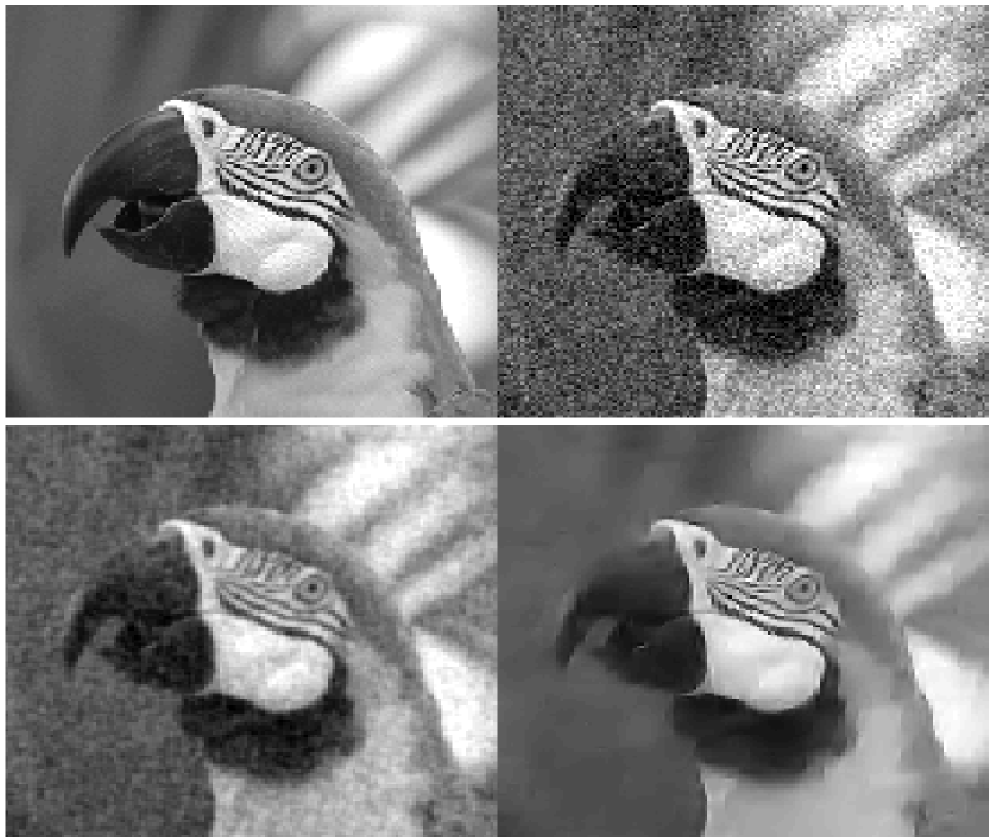


Gabor wavelets,
edge detection,
receptive fields of V1 cells...

Image denoising

Original image

Noisy image



Wiener filtering

ICA filtering