Fast Algorithms for Computing Tree LCS

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Definition of tree LCS

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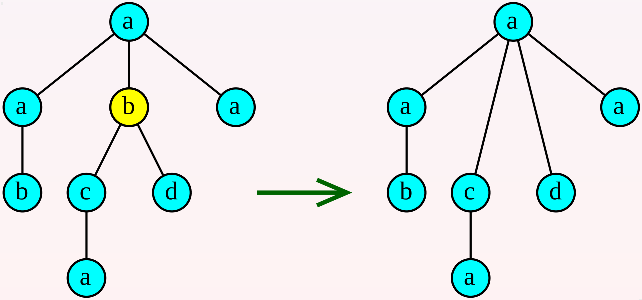
The LCS of two trees is the size of largest forest that can be obtained from both trees by deleting nodes.

Deletion of a node v in a forest F means that the children of v become children of the parent of v (if it exists).



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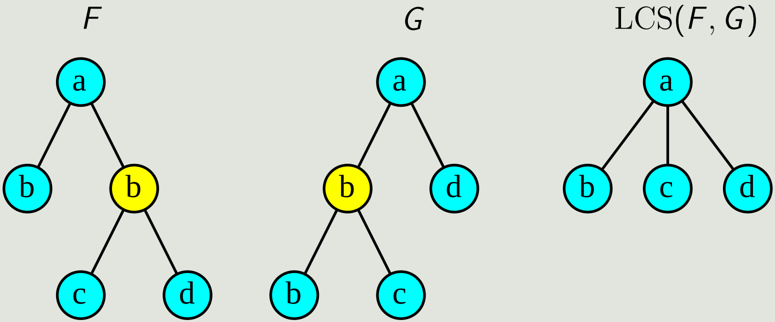
Example



Known & new results

Let F and G be the input trees, with sizes $n \geq m$.
Known algorithms for tree edit distance/tree LCS:

- $O(nm \cdot \mathrm{leaves}(F)^2 \cdot \mathrm{leaves}(G)^2) = O(n^6)$ (Tai 79)
- $O(nm \cdot \mathrm{cdepth}(F) \cdot \mathrm{cdepth}(G)) = O(n^4)$ (Zhang & Shasha 89)
- $O(m^2 n \log n) = O(n^3 \log n)$ (Klein 98)
- $O(nm^2(1 + \log \frac{n}{m})) = O(n^3)$ (Demaine et al. 07)

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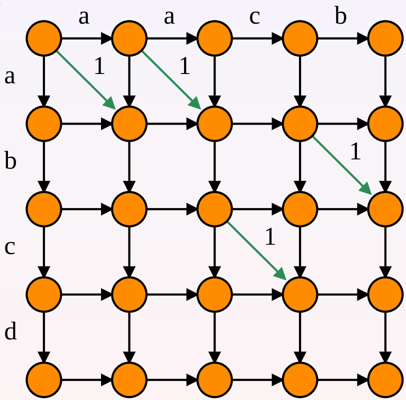
New algorithms for tree LCS:

- $O(r \cdot \text{height}(F) \cdot \text{height}(G) \cdot \log\log m)$
- $O(Lr \log r \cdot \log\log m)$

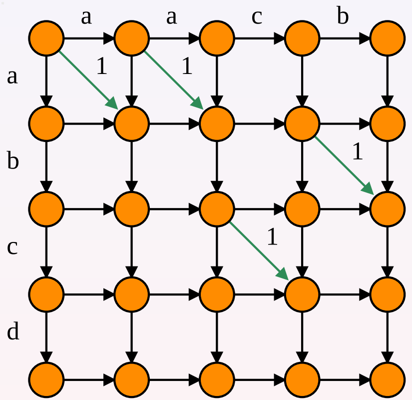
$r =$ number of match pairs (pairs of vertices $v \in F$, $w \in G$ with equal labels)
$L = \text{LCS}(F, G)$

Alignment graph for string LCS



- Let S and T be strings of lengths n and m.
- The alignment graph has vertices (i, j) for every $0 \leq i \leq n$ and $0 \leq j \leq m$. Vertex (i, j) corresponds to the pair of prefixes $S[1..i]$, $T[1..j]$.

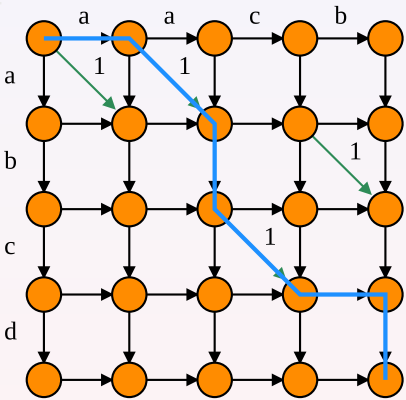
Alignment graph for string LCS



The edges are

- Vertical and hotizontal edges of weight 0 corresponding to deletion of character from S or T.
- Diagonal edges for every match pair of weight 1.

Alignment graph for string LCS



- A path of weight k from $(0,0)$ to (n,m) corresponds to a common substring of size k.
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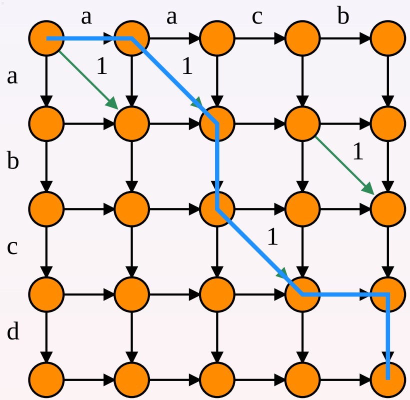
Alignment graph for string LCS

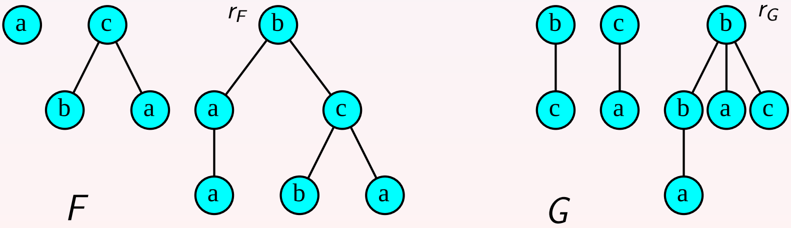


- The number of nonzero weight edges is r.
- The maximum weight path can be found in time $O(r \log \log m)$ (Hunt-Szymanski 77).

The algorithm of Zhang & Shasha

R_F = rightmost tree in F. r_G = root of R_G.

$$\mathrm{LCS}(F,G) = \max \begin{cases} \mathrm{LCS}(F - r_F, G), \\ \mathrm{LCS}(F, G - r_G), \\ \mathrm{LCS}(R_F - r_F, R_G - r_G) \\ \quad + \mathrm{LCS}(F - R_F, G - R_G) + 1 \\ \text{if } \mathrm{label}(r_F) = \mathrm{label}(r_G) \end{cases}$$



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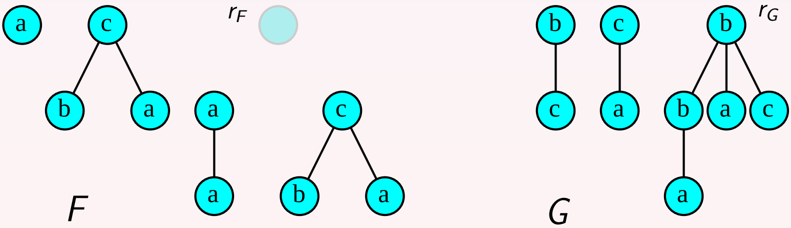

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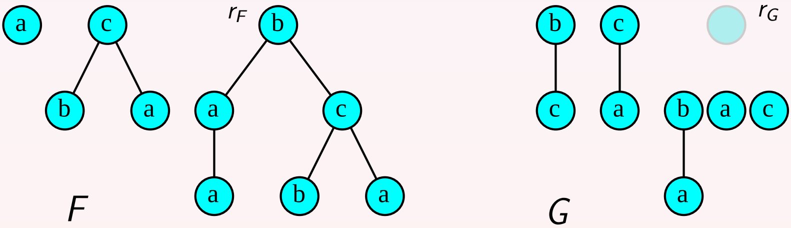

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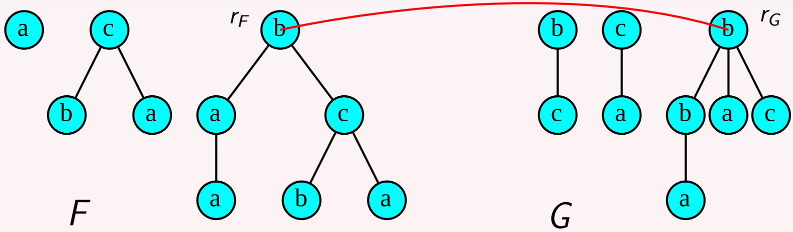

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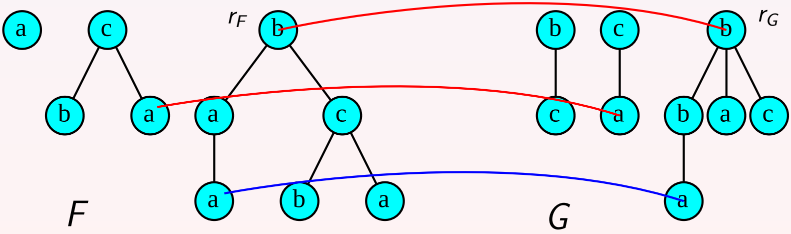

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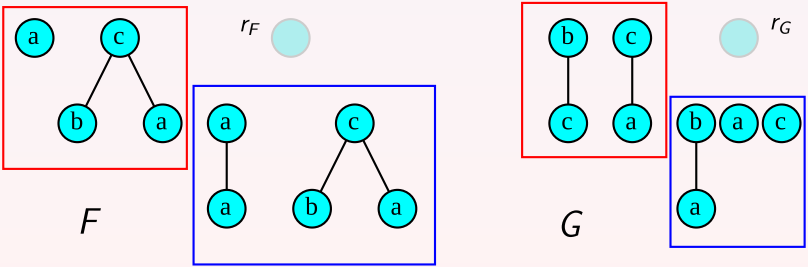

Relevant subproblems

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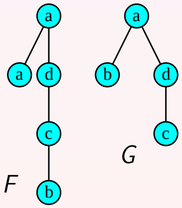

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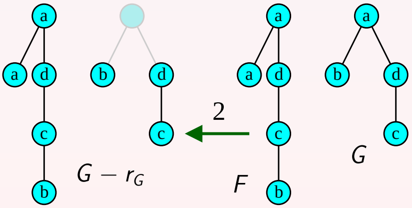

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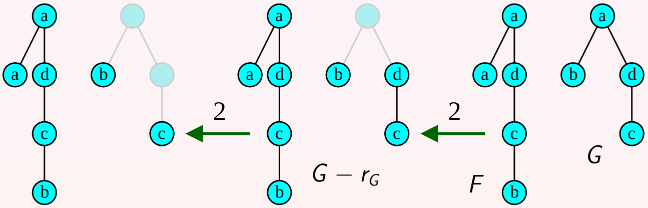

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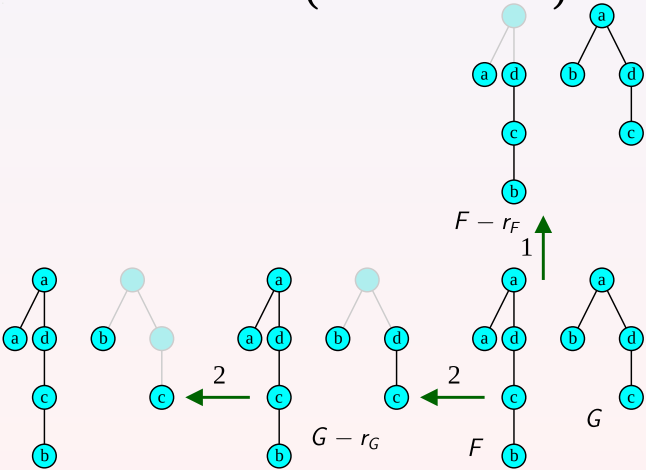

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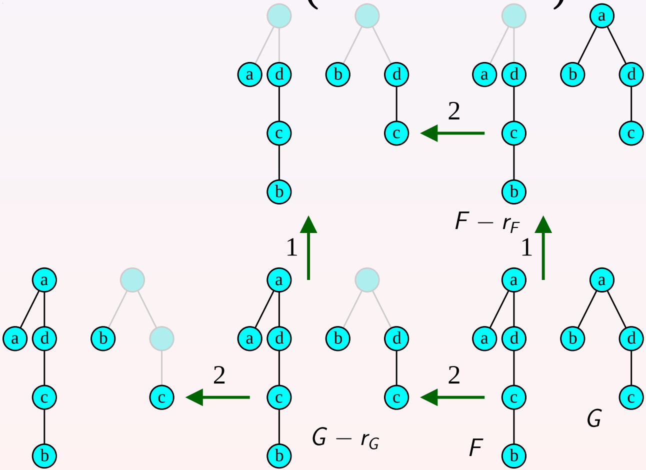

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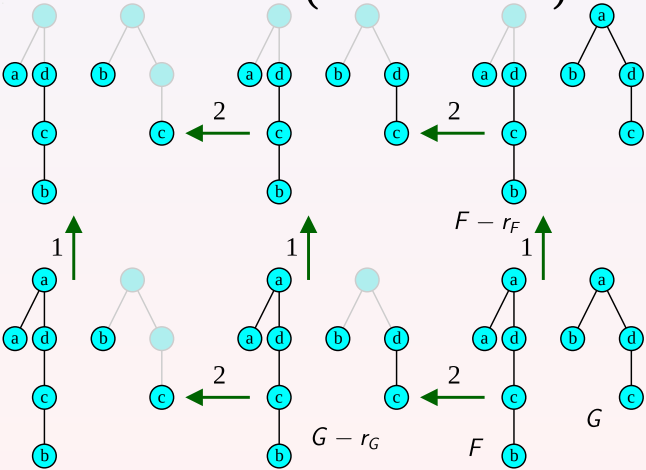

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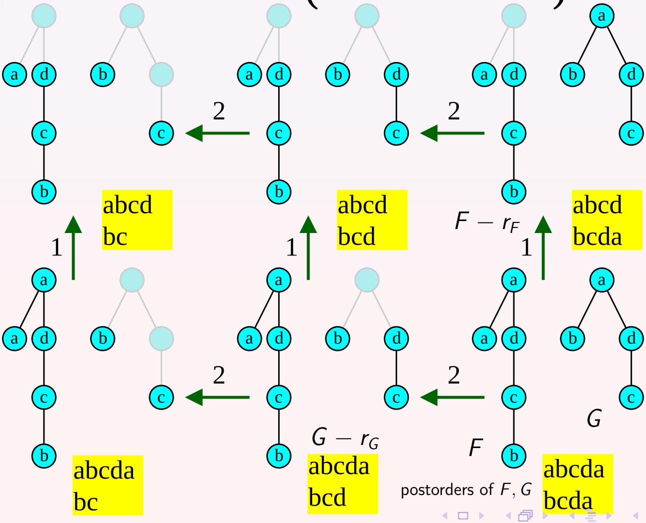
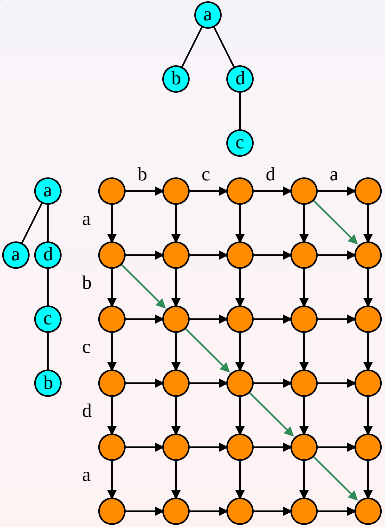


Alignment graph (Backofen et al., Touzet)



Consider the alignment graph for the two postorder strings of F and G.

A path of weight k does not correspond to a common subforest of size k.

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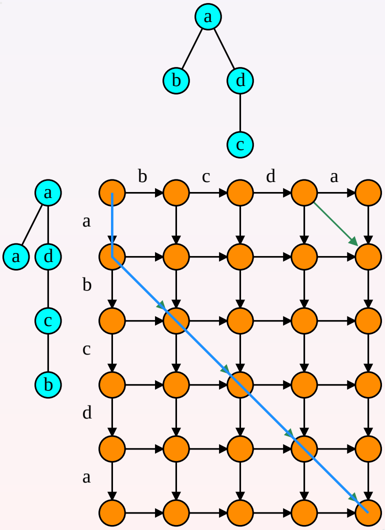

Consider the alignment graph for the two postorder strings of F and G.

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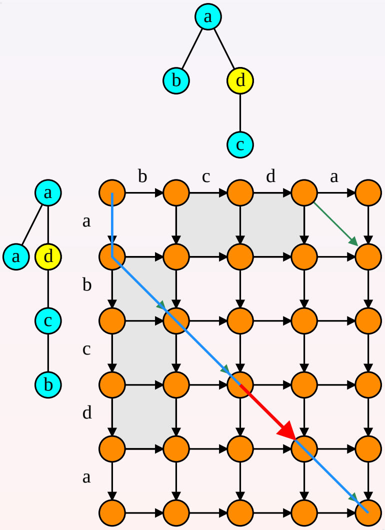

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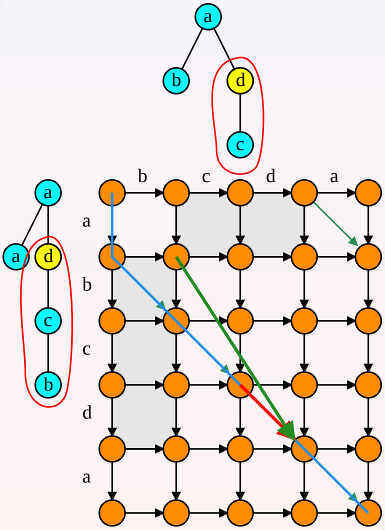

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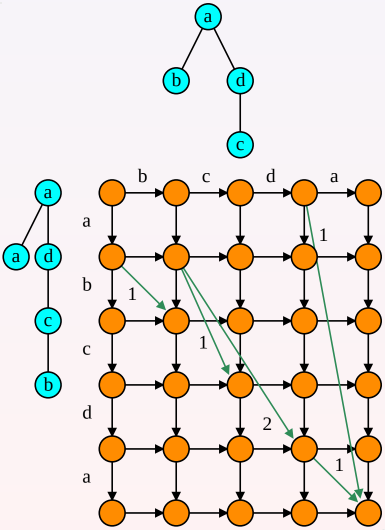


For each match pair v, w (with $v \in F$, $w \in G$) add an edge

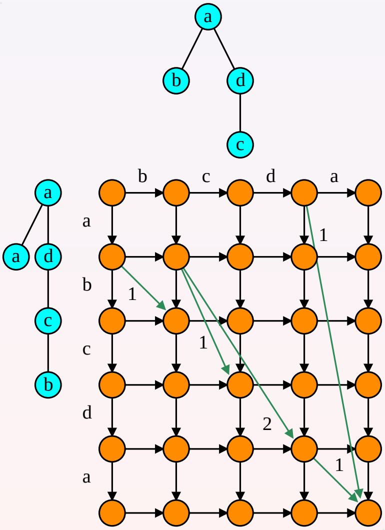
$$(i - |F_v|, j - |G_w|) \rightarrow (i, j)$$

of weight $\mathrm{LCS}(F_v, G_w)$, where i and j are the ranks of v and w in the postorders of F and G.

For the match pair of the roots we just add an edge $(n-1, m-1) \rightarrow (n, m)$ of weight 1.

The LCS values are computed recursively.

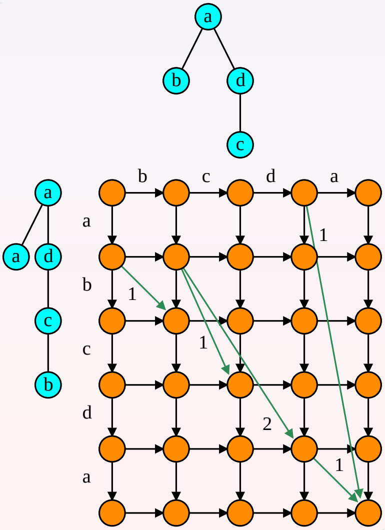
Alignment graph (Backofen et al., Touzet)



Every path of weight k from $(0,0)$ to (n,m) corresponds to a common subforest of size k.

Every largest common subforest corresponds to a path of weight $\mathrm{LCS}(F,G)$.

Alignment graph (Backofen et al., Touzet)



There are exactly r edges with nonzero weight.

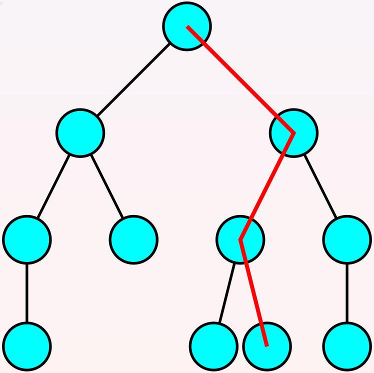
Computing the maximum weight path in the graph takes $O(r \log \log m)$ time.

The time including the recursive calls is $O(r \cdot \text{height}(F) \cdot \text{height}(G) \cdot \log \log m)$

Heavy path

Definition

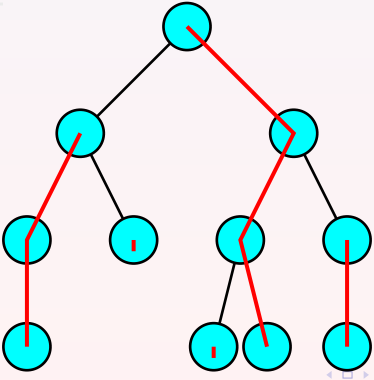
A heavy path of a tree is a path that starts at the root and goes from each vertex v to the child of v whose subtree contains more vertices.



Heavy path decomposition

Definition

A heavy path decomposition of a tree F is a collection of paths obtained by taking a heavy path of F, removing its vertices, and then building a heavy path decomposition for every remaining tree.



The algorithm of Klein

R_S = rightmost tree in S. r_S = root of R_S.

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L_S = leftmost tree in S. l_S = root of L_S.

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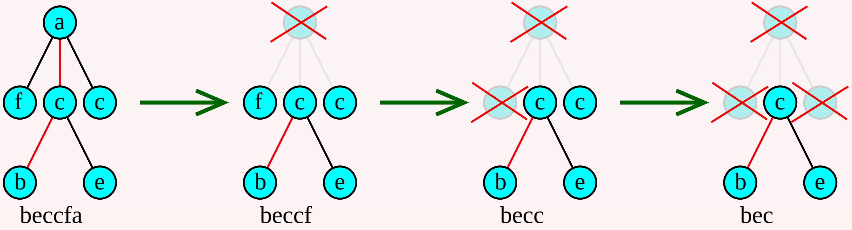
The algorithm of Klein

- Let F and G be the input trees, with $|F| \geq |G|$.
- Build a heavy path decomposition of F.
- Recursively compute $\mathrm{LCS}(F, G)$ using left & right rules.
- To compute $\mathrm{LCS}(F', G')$ for some F' and G', let p be the highest path in decomposition that contain vertices of F'. Use the left rule if l_F is not on p. O/w use left rule.



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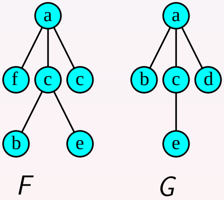

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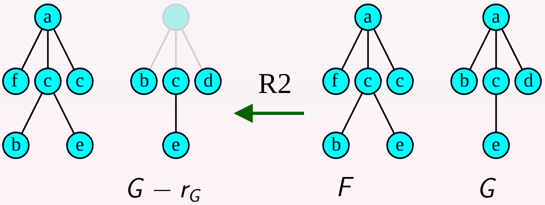

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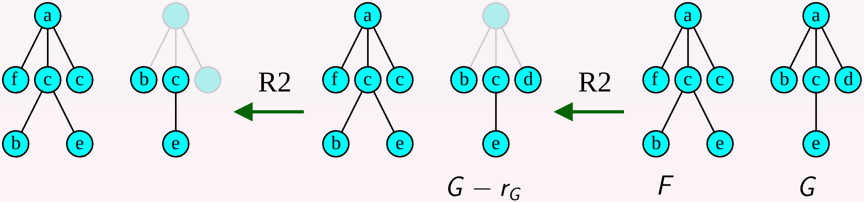

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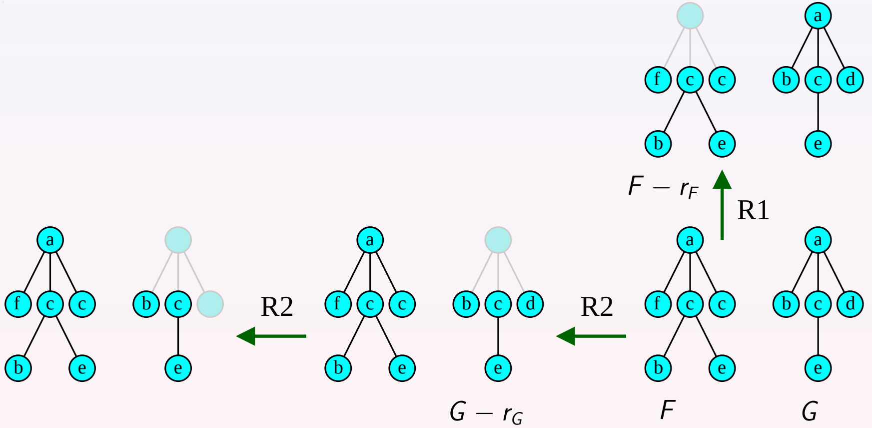

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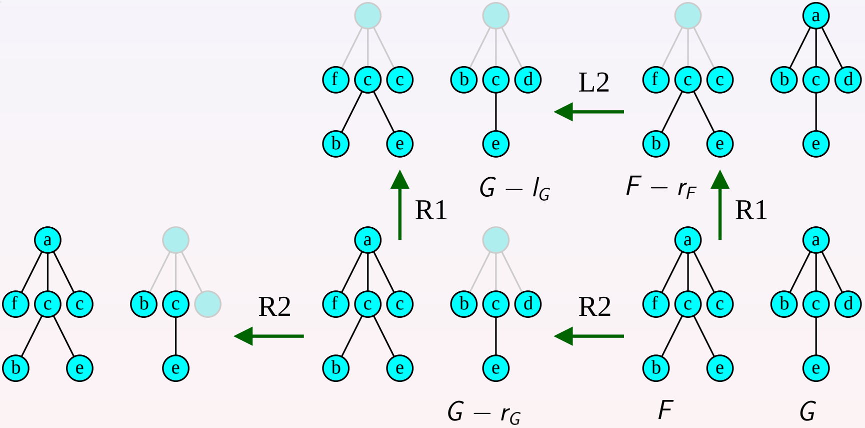

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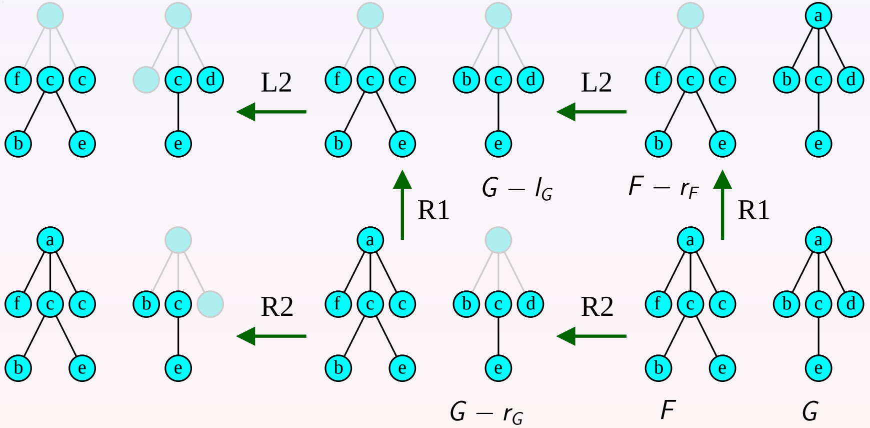
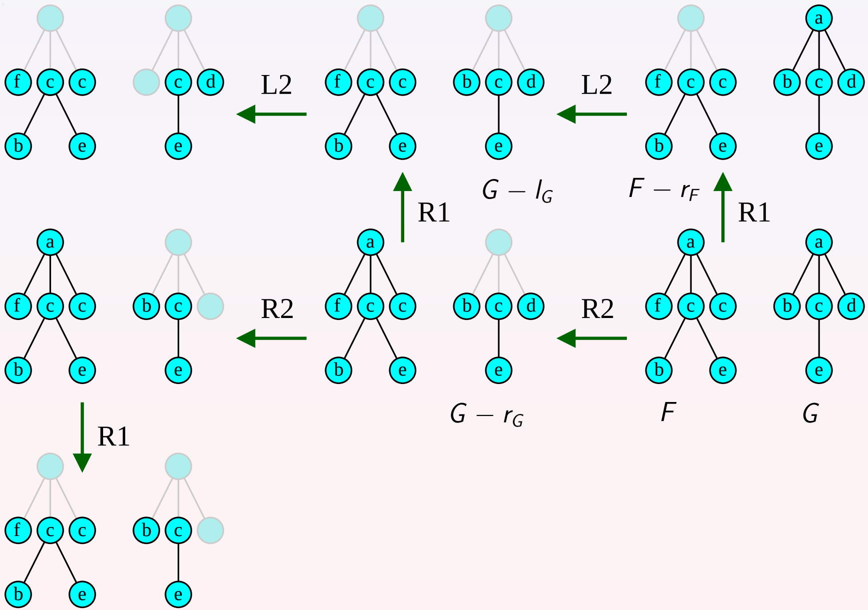
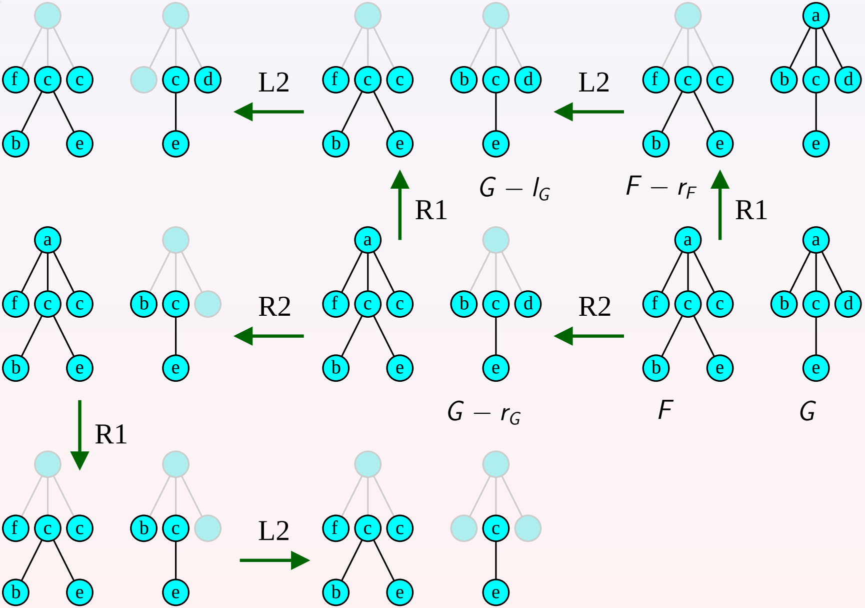


$$\mathrm{LCS}(F,G) = \min \begin{cases} \mathrm{LCS}(F - l_F, G), & L1 \\ \mathrm{LCS}(F, G - l_G) & L2 \\ \dots \end{cases} \qquad \mathrm{LCS}(F,G) = \min \begin{cases} \mathrm{LCS}(F - r_F, G), & R1 \\ \mathrm{LCS}(F, G - r_G) & R2 \\ \dots \end{cases}$$

Relevant subproblems

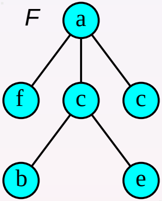


Relevant subproblems

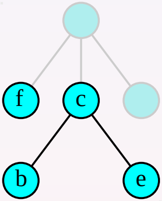


Relevant subproblems

- F represented by a string S_F according to deletion order.

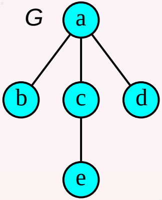


$S_F = \text{beccfa}$

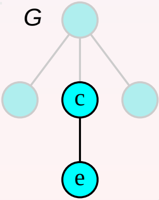


$S_F[1..4] = \text{becc}$

- G represented by its Euler string S_G.



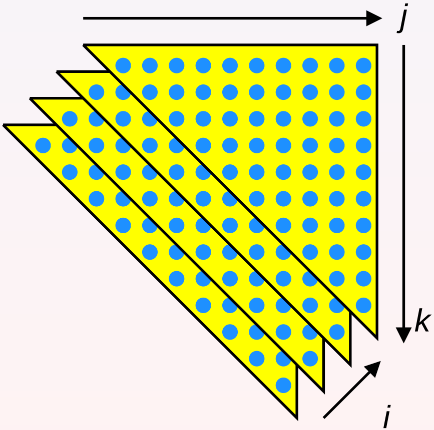
$S_G = \text{abbceecdda}$



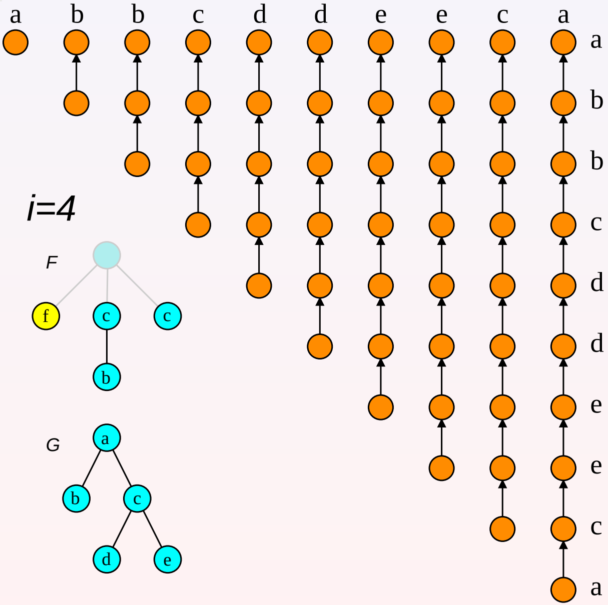
$S_G[3..7] = \text{bceec}$

Alignment graph

The alignment graph has vertex (i, j, k) for every $0 \leq i \leq n$ and $1 \leq j \leq k \leq 2m$.
Vertex (i, j, k) corresponds to the pair $S_F[1..i], S_G[j..k]$



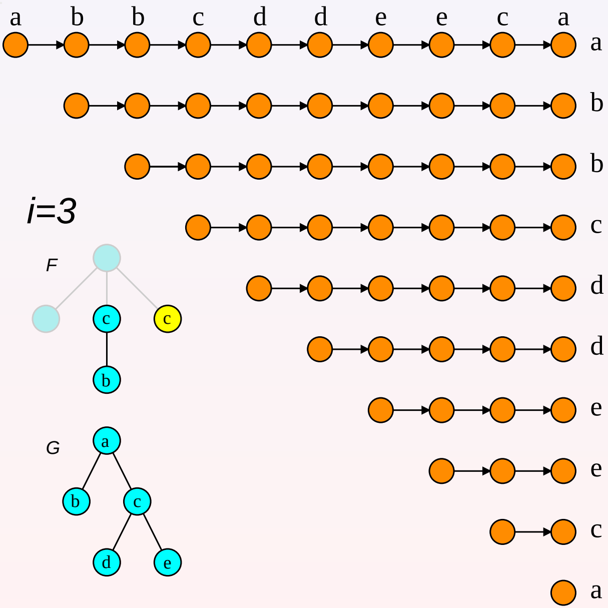
Alignment graph



Edges corresponding to deleting leftmost root of $S_G[j..k]$.

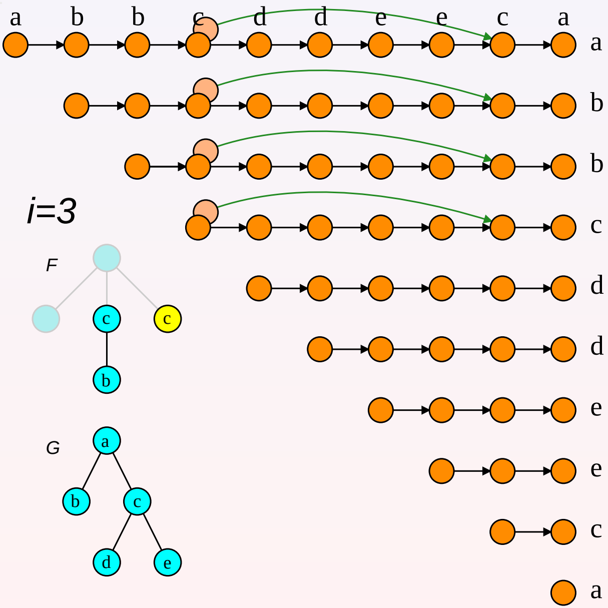
Weights: 0

Alignment graph



Edges corresponding to deleting rightmost root of $S_G[j..k]$.
Weights: 0

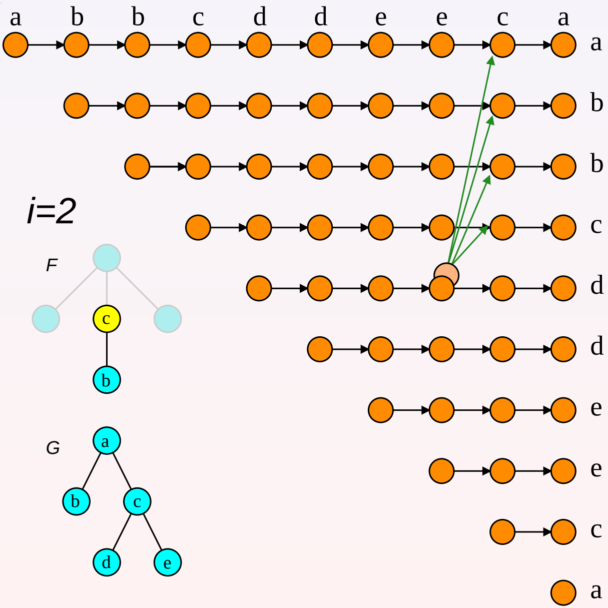
Alignment graph



Edges corresponding to matching rightmost roots of $S_F[1..i]$ and $S_G[j..k]$.

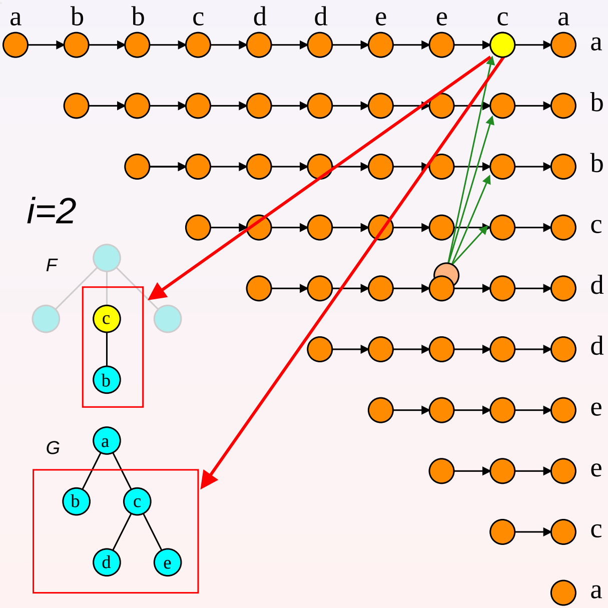
Weights: ≥ 1 (LCS)

Alignment graph



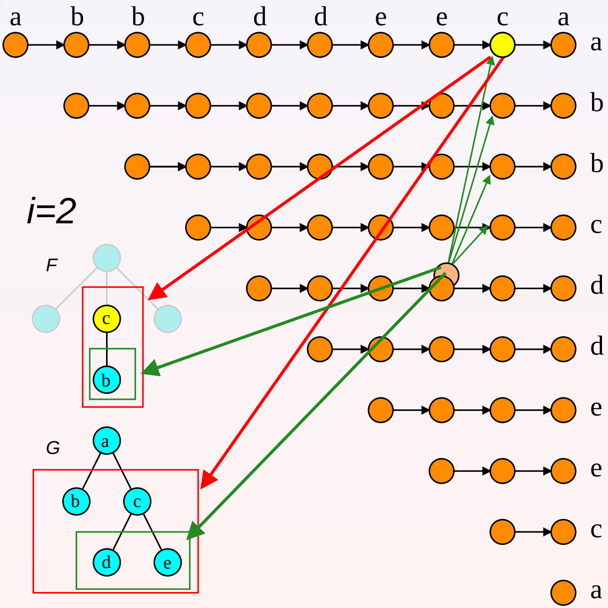
If $S_F[1..i]$ is a tree, the edges are different.
Weights: 1

Alignment graph



If $S_F[1..i]$ is a tree, the edges are different.
Weights: 1

Alignment graph



If $S_F[1..i]$ is a tree, the edges are different.
Weights: 1

Summary

Algorithms for tree LCS:

- $O(r \cdot \mathrm{height}(F) \cdot \mathrm{height}(G) \cdot \log\log m)$
- $O(Lr \log r \cdot \log\log m)$

Algorithm for homeomorphic tree LCS

- $O(r \lg\lg m \cdot (\mathrm{height}(F) + \mathrm{height}(G)))$.