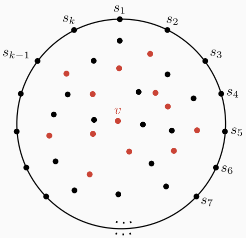
Improved Compression of the Okamura-Seymour Metric

Shay Mozes, Nathan Wallheimer, Oren Weimann

The Okamura-Seymour Metric Compression Problem



- An undirected, unweighted planar graph $G = (V, E)$.
- A set $S = \{s_1, s_2, \ldots, s_k\}$ of k consecutive vertices on a face f_∞.
- A set $T \subseteq V$ of terminal vertices lying anywhere in the graph.

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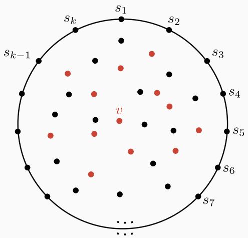

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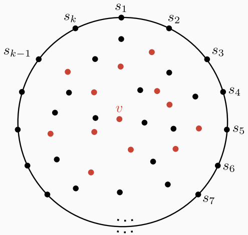
The Okamura-Seymour Metric Compression Problem

Succinctly encode the $T \times S$ distances to answer $d(v, s_i)$ queries.

Results



	Solution	Complexity
$T = S$	Unit-Monge [AGMW'18]	$\tilde{O}(k)$ space, $\tilde{O}(1)$ query
$T = V$	MSSP [Klein'05]	$O(n)$ space, $\tilde{O}(1)$ query
$T \subset V$	Naïve	$\tilde{O}(\lvert T \rvert \cdot k)$ space, $O(1)$ query

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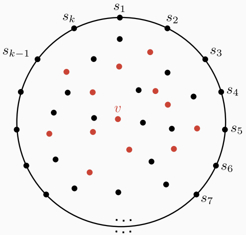

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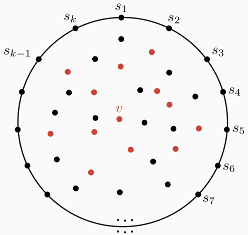

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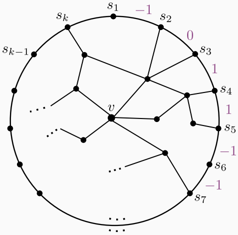
The Pattern of $v \in V$



$$p_v = \langle d(v, s_2) - d(v, s_1), d(v, s_3) - d(v, s_2), \cdots, d(v, s_k) - d(v, s_{k-1}) \rangle$$

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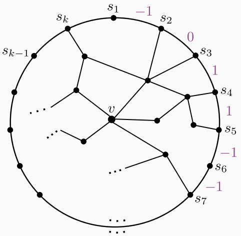

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- $p_v \in \{-1, 0, 1\}^{k-1}$ by the triangle inequality.
- v-to-s_i distances are determined by p_v and $d(v, s_1)$:

$$d(v, s_i) = d(v, s_1) + \underbrace{\sum_{j=1}^{i-1} p_v[j]}_{\text{prefix-sum}}$$

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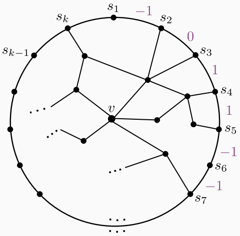


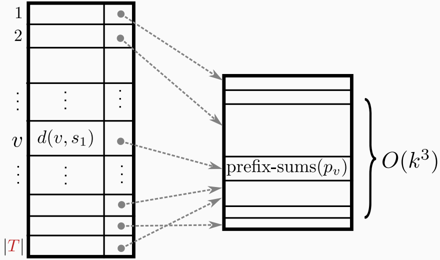
Theorem (Li & Parter [STOC 2019])

There are only $x = O(k^3)$ distinct patterns among all vertices of the graph.

Huge improvement over the trivial $O(3^k)$ bound.

Li & Parter's Compression

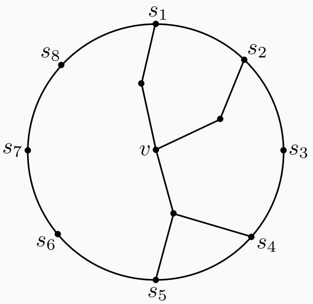
1. One table with the $O(k^3)$ distinct patterns and their prefix-sums.
2. Every $v \in T$ stores $d(v, s_1)$ and a pointer to p_v in the previous table.



Space: $\tilde{O}(|T| + k^4)$, Query time: $O(1)$.

Li & Parter's $O(k^3)$ Proof

Assume w.l.o.g. that the patterns are over $\{-1, 1\}$ and not $\{-1, 0, 1\}$. This can be achieved by subdividing every edge:



Li & Parter's $O(k^3)$ Proof

Arrange the n binary patterns as the rows of a binary matrix.
By planarity, there are no four columns $a < b < c < d$ such that for some $u, v \in V$:

$$\begin{array}{c} \\ \vdots \\ u \\ \vdots \\ v \\ \vdots \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} & \vdots & \vdots & \\ -1 & 1 & -1 & 1 \\ & \vdots & \vdots & \\ 1 & -1 & 1 & -1 \\ & \vdots & \vdots & \end{pmatrix} \end{array}$$

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Thus, the VC-dimension of the matrix is at most 3.
By the Sauer-Shelach Lemma, the number of distinct rows is $O(k^3)$.

Our Results

Let $x =$ the number of distinct patterns among all vertices of G.

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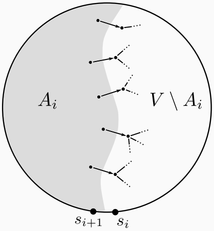
Bisectors



For every $1 \leq i < k$, define the following cuts:

$$A_i = \{v \in V \mid p_v[i] = -1\}$$
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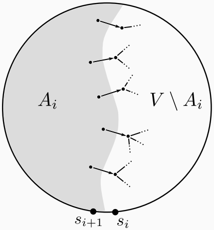

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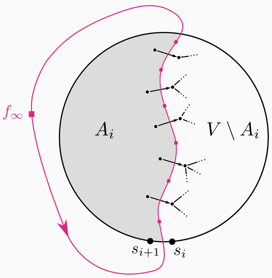

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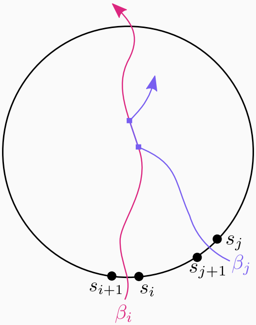
β_i **is a directed simple cycle in the dual graph.**

Every Two Bisectors are Arc-Disjoint



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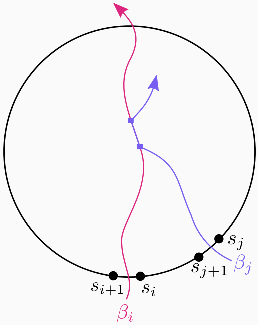

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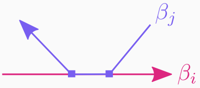
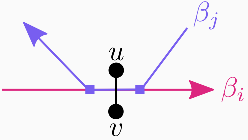
However, it is possible that β_i contains *reversed* arcs of β_j:



Patterns of Adjacent Vertices Differ in at Most Two Bits

For any $\{u, v\} \in E(G)$, u and v are separated by at most two bisectors.



$$\langle 1, -1, \cdots, -1, \cdots, 1, \cdots - 1, -1 \rangle$$

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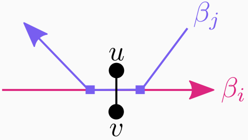

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Hence, p_u and p_v differ in at most two bits.

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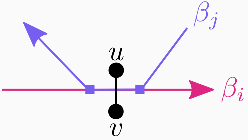

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Hence, p_u and p_v differ in at most two bits.

We can use this fact to get a compression of size $\tilde{O}(|T| + x + k)$

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Let x = the number of distinct patterns among all vertices of G.

- **An $\tilde{O}(|T| + x + k)$ bits compression of the Okamura-Seymour metric, with query time $\tilde{O}(1)$.**
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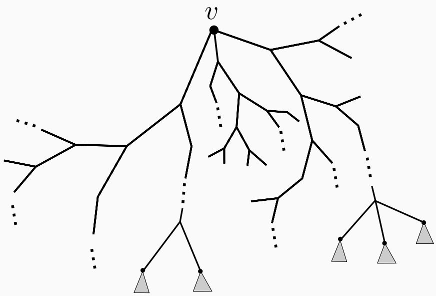
A Spanning Tree of G



- Root the tree at an arbitrary vertex v.

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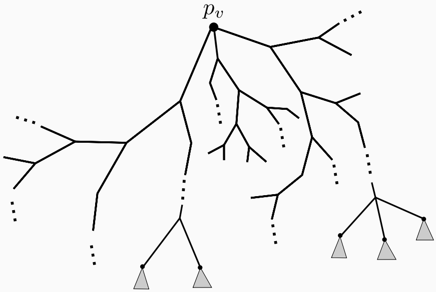

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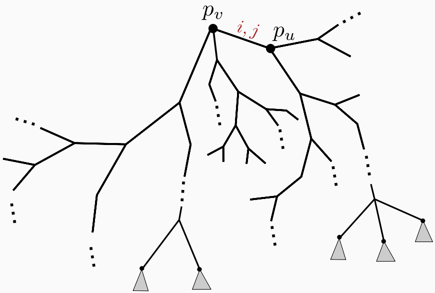
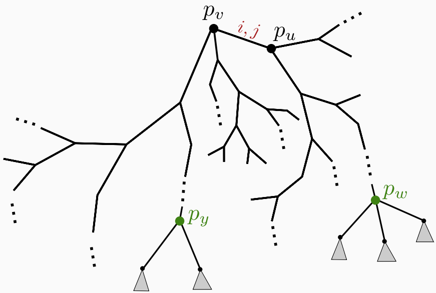


- Root the tree at an arbitrary vertex v.
- Identify each node with its pattern.
- Label every edge by the two bits that change.

Some Patterns Appear Multiple Times



1. Let y and w be two vertices such that $p_y = p_w$.

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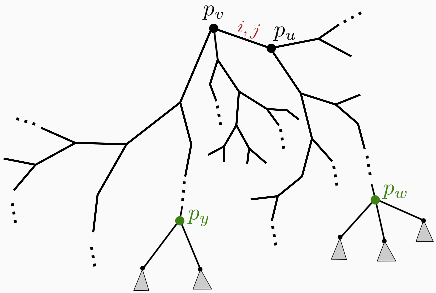

1. Let y and w be two vertices such that $p_y = p_w$.
2. Assume w.l.o.g. that $depth(y) \leq depth(w)$.

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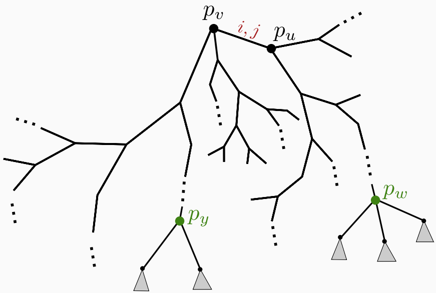

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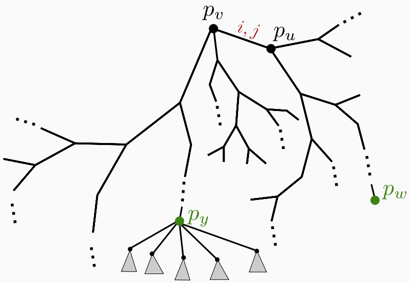

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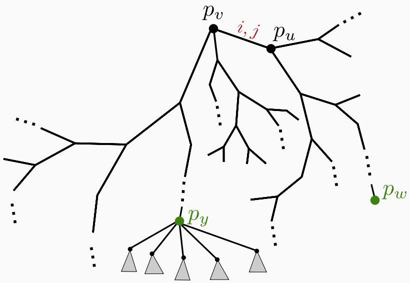

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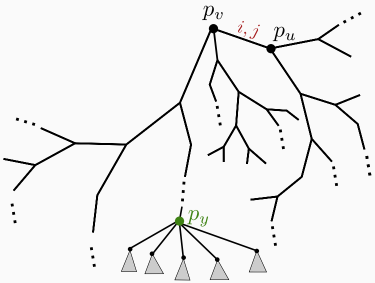

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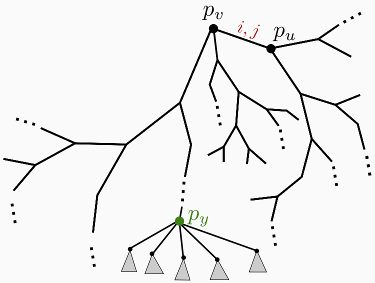

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4. Remove w from the tree.
5. Repeat until the size of the tree is x.

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In total: $\tilde{O}(|T| + x + k)$ space and $\tilde{O}(1)$ time for query.
Preprocessing: can be done in $\tilde{O}(n)$ time.

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- **An optimal $\tilde{O}(|T| + k)$ bits compression with $\tilde{O}(1)$ query for two special cases: (1) T induces a connected subgraph of G, and (2) T lies on a single face.**
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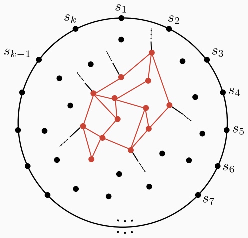

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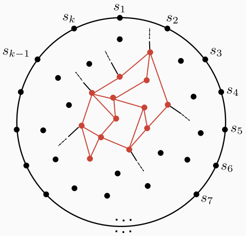

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Space: $\tilde{O}(|T| + k)$. Time: $\tilde{O}(1)$.

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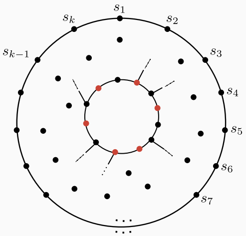

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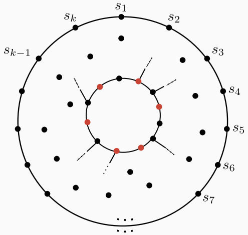

We only need to traverse the vertices along the face. Every bisector can visit the face at most once. Thus, there are only $x = O(k)$ versions.

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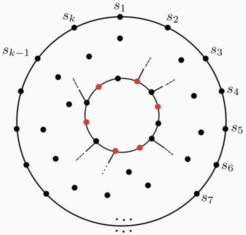

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Space: $\tilde{O}(|T| + k)$. Time: $\tilde{O}(1)$.

Our Results

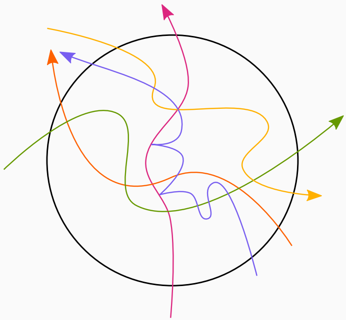
Let x = the number of distinct patterns among all vertices of G.

- An $\tilde{O}(|T| + x + k)$ bits compression of the Okamura-Seymour metric, with query time $\tilde{O}(1)$.
- An optimal $\tilde{O}(|T| + k)$ bits compression with $\tilde{O}(1)$ query for two special cases: (1) T induces a *connected* subgraph of G, and (2) T lies on a single face.
- **An alternative $x = O(k^3)$ proof that exploits planarity beyond VC-dimension. Namely, planar duality and the fact that distances among vertices of S are bounded by k.**
- In Halin graphs, we show that $x = \Theta(k^2)$ while the VC-dimension argument is limited to showing $O(k^3)$.

The Bisector Graph $G_{\mathcal{B}}$

Definition

The *bisector graph* $G_{\mathcal{B}}$, is composed of the union of all the bisectors.



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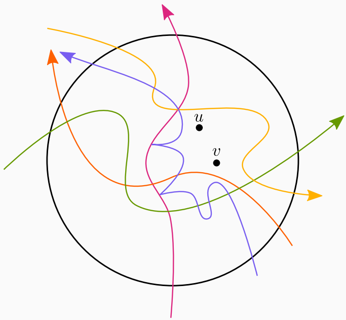

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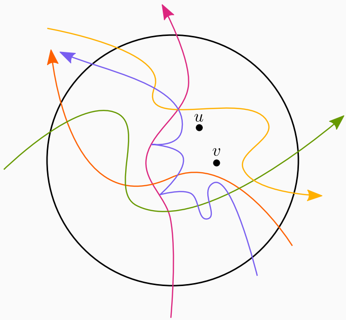

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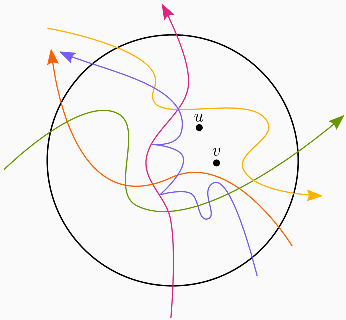

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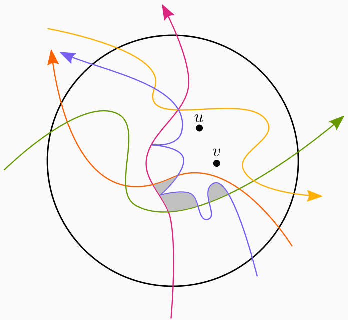

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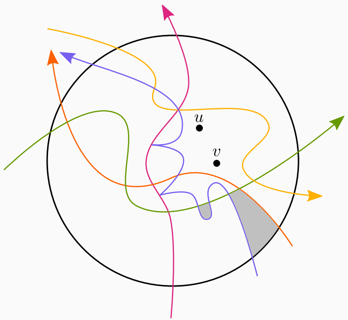

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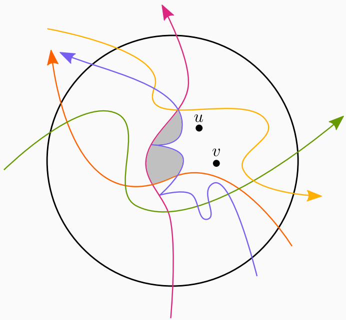

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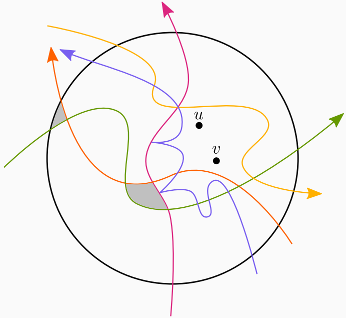

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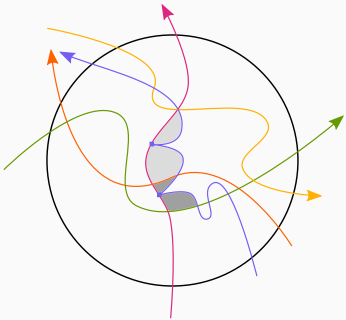
Remove Touching Points

A *touching* is an intersection of bisectors without *crossing*.



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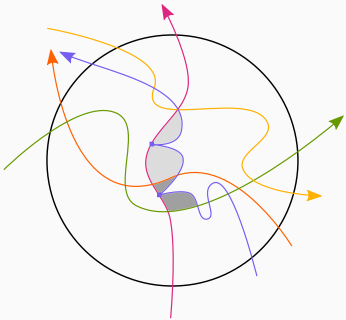

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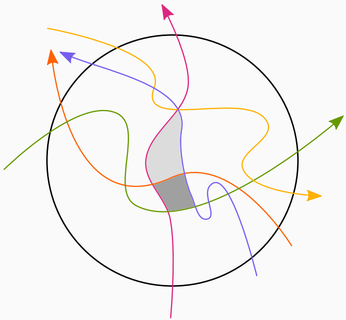

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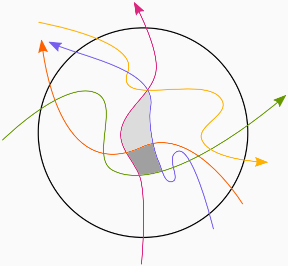

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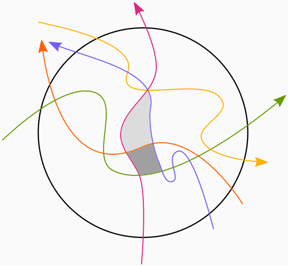

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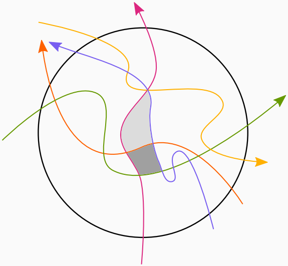

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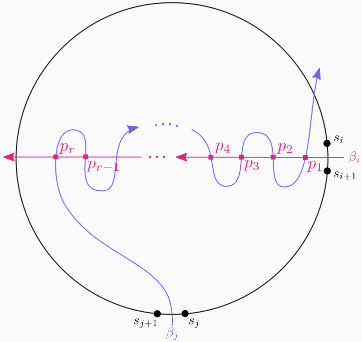
Our main technical contribution: every two bisectors can cross at most $O(k)$ times, hence the number of crossings is $O(k^3)$.

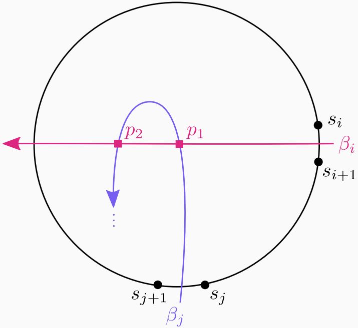
Every Two Bisectors Cross in Opposite Orientation



Lemma

Let $p_1, p_2, \ldots, p_r$ be the crossing points of β_i and β_j, in the order they appear along β_i.
The crossing points along β_j are reversed $p_r, p_{r-1}, \ldots, p_1$.

Proof by Contradiction



- Assume for contradition that p_1 appears before p_2 in β_j.

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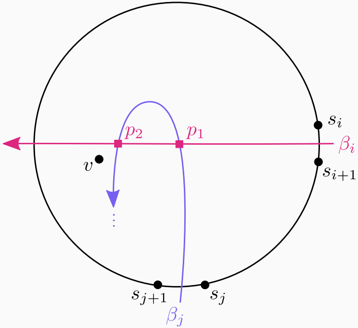

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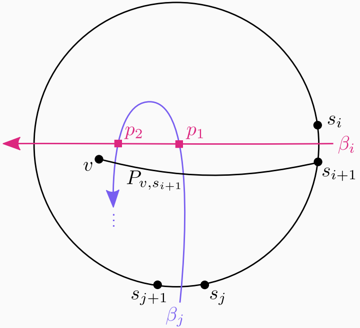

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- $P_{v,s_{i+1}}$ must stay on the left of β_i and cross β_j.

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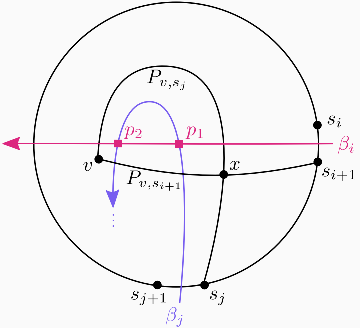

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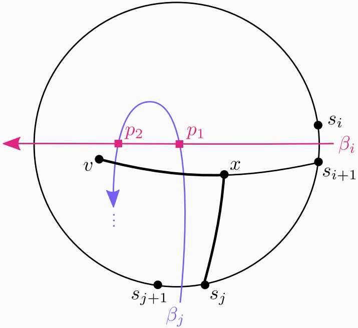

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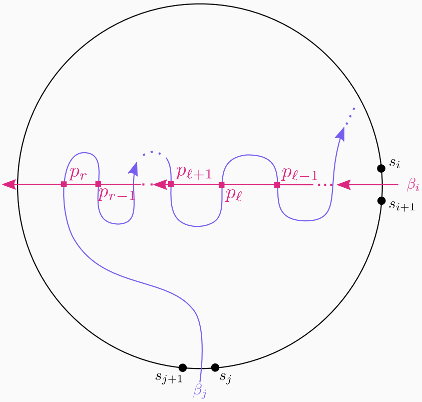
- **There's a shortest v-to-s_j path that crosses β_j, a contradiction.**

Two Bisectors can Cross at Most $O(k)$ Times



Lemma

The number of crossings between β_i and β_j is $r = O(k)$.

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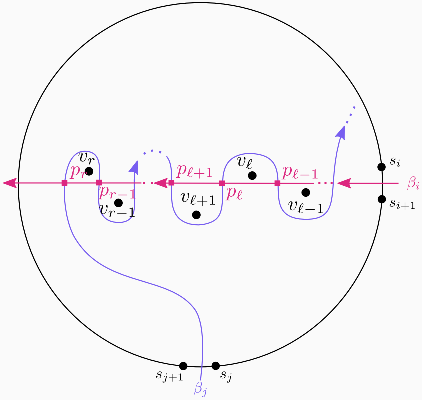

- Let $v_1, v_2, \ldots, v_r$ be primal vertices inside the "pockets" created between consecutive crossings.

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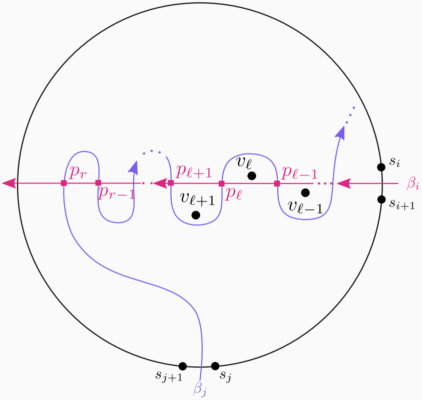

- Let $v_1, v_2, \ldots, v_r$ be primal vertices inside the "pockets" created between consecutive crossings.
- Consider some v_ℓ.

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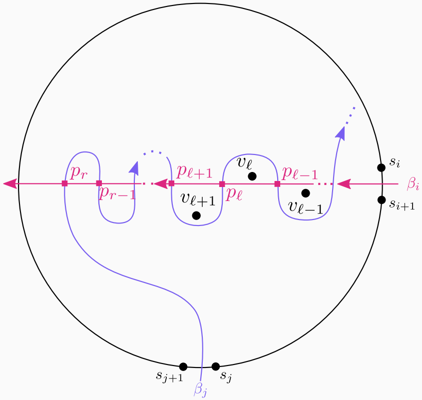

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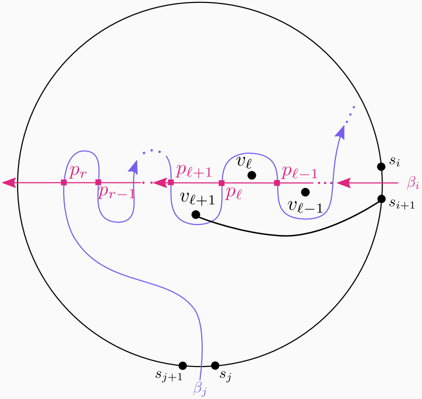

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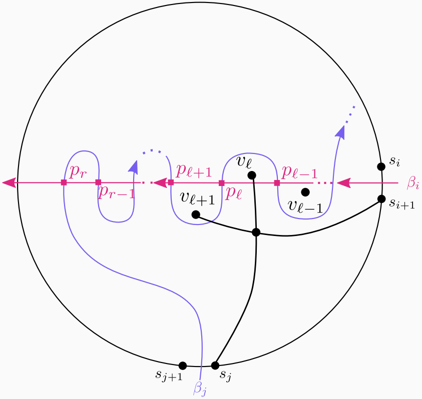

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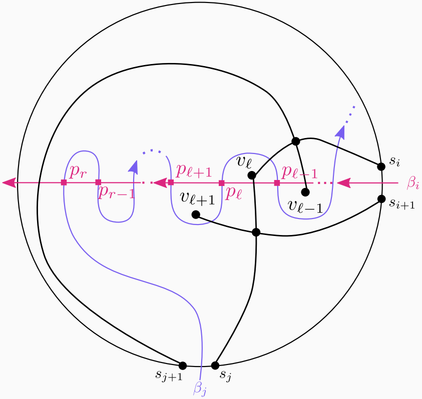

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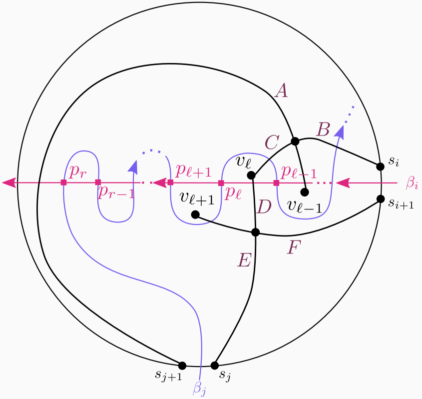

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- A symmetric configuration with v_ℓ and $v_{\ell-1}$.
- Denote the lengths of subpaths by A, B, C, D, E, F.

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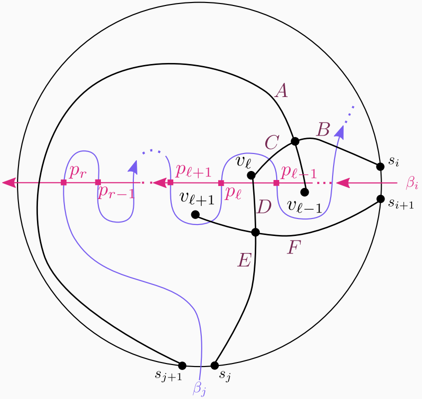

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$$C + B \leq D + F - 1$$

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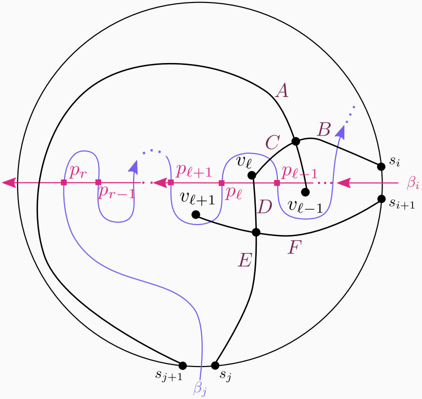

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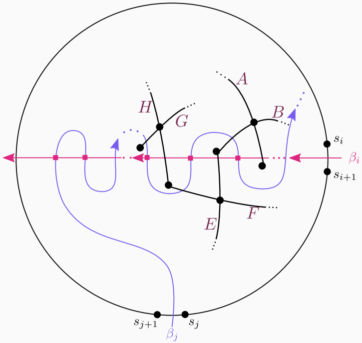

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and so on...

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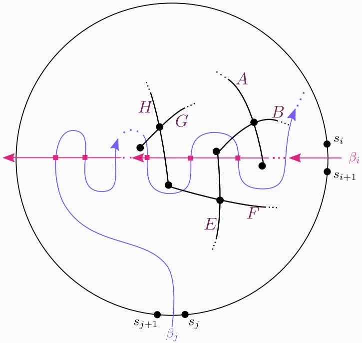

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$$\Omega(\boldsymbol{r}) \leq \boldsymbol{d}(\boldsymbol{v}, \boldsymbol{s_i}) - \boldsymbol{d}(\boldsymbol{v}, \boldsymbol{s_{j+1}}) \leq \boldsymbol{k}$$

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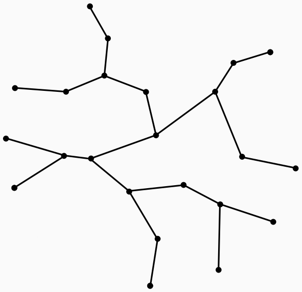
Halin Graphs



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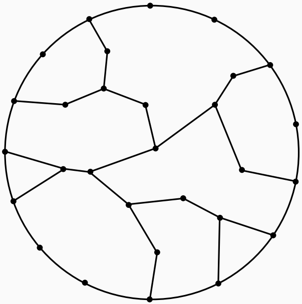

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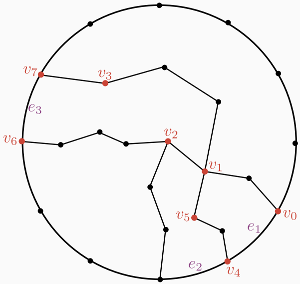
Denote by k the size of the cycle. Consider the patterns of the graph w.r.t. the infinite face.

A Limitation of the VC-dimension Argument



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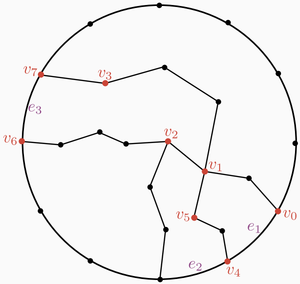

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$$\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \end{array}$$

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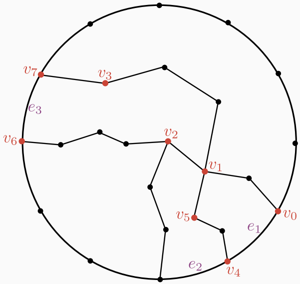

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$$

The VC-dimension of the matrix is 3

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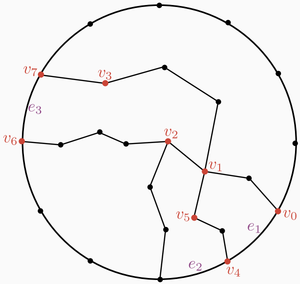


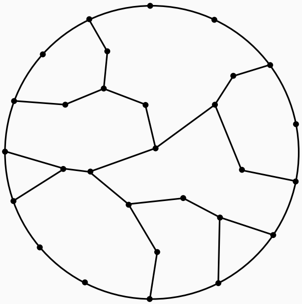
Consider the values of the patterns of $v_0, v_1, \dots v_7$ at e_1, e_2, e_3:

	e_1	e_2	e_3
v_0	1	1	1
v_1	1	1	−1
v_2	1	−1	1
v_3	1	−1	−1
v_4	−1	1	1
v_5	−1	1	−1
v_6	−1	−1	1
v_7	−1	−1	−1

The VC-dimension of the matrix is 3

Thus, the VC-dimension argument is limited to showing $O(k^3)$.

An $O(k^2)$ Proof in Halin Graphs



- Subdivide every edge to make patterns binary.
- There are only $O(k)$ vertices of degree > 2, hence $O(k)$ faces in G.
- Every bisector can visit every face at most once.
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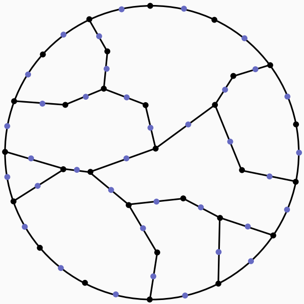

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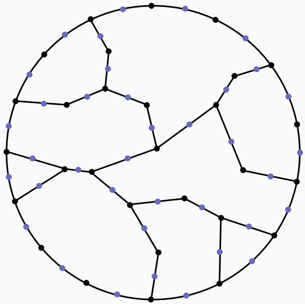

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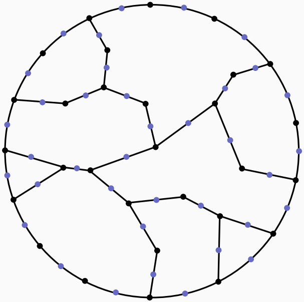

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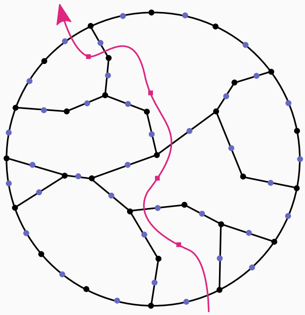

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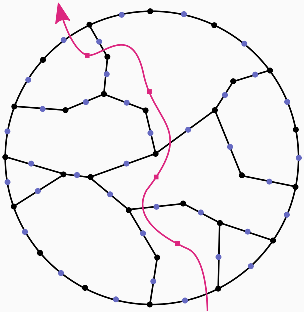

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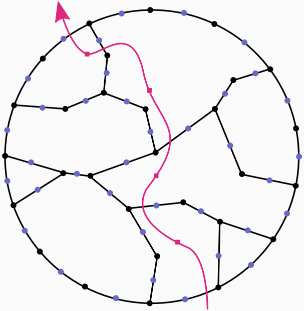

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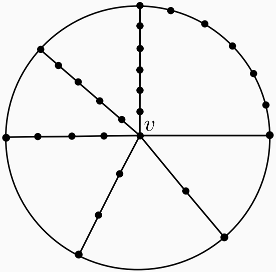
A matching $\Omega(k^2)$ Lower Bound for Halin Graphs



- Attach $O(k)$ paths of lengths $1, 2, \ldots, \frac{k}{2}$ to a middle vertex v.
- Pad the portions of the infinite face between the first and last paths with $\frac{k}{2}$ vertices.

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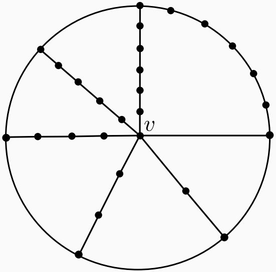

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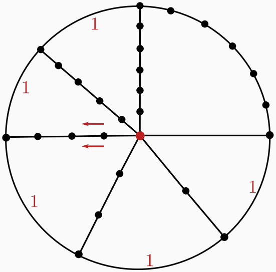
Claim: The path of length i contains $i - 1$ distinct patterns.

The Different Patterns Along a Path



$\langle 1, 1, 1, 1, 1 \rangle$

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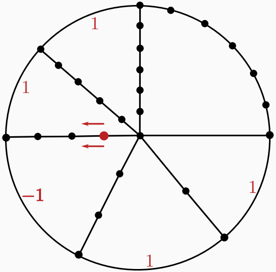

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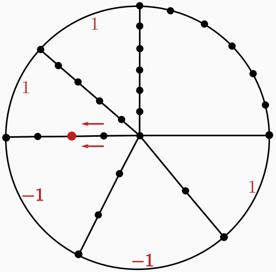

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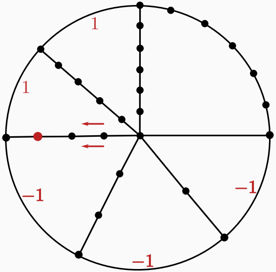
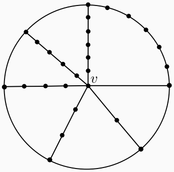


$$\langle 1, 1, 1, 1, 1\rangle, \langle 1, 1, -1, 1, 1\rangle, \langle 1, -1, -1, 1, 1\rangle, \langle -1, -1, -1, 1, 1\rangle$$

A matching $\Omega(k^2)$ Lower Bound for Halin Graphs



The distinct patterns of this graph are thus:

$\langle -1, 1, 1, 1, 1, \ldots \rangle$

$\langle 1, -1, 1, 1, 1, \ldots \rangle, \langle -1, -1, 1, 1, 1, \ldots \rangle$

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$$\vdots$$

$$\sum_{i=1}^{\frac{k}{2}} (i-1) = \Omega(k^2)$$

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Conjecture

The number of distinct patterns in a planar graph is $O(k^2)$.

Open Question 2: Information-Theoretic Lower Bounds

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