Introduction to Number Theory 1

Division

Definition: Let a and b be integers. We say that **a divides b**, or **$a|b$** if $\exists d$ s.t. $b = ad$. If $b \neq 0$ then $|a| \leq |b|$.

Division Theorem: For any integer a and any positive integer n, there are unique integers q and r such that $0 \leq r < n$ and $a = qn + r$.

The value $r = a \bmod n$ is called the **remainder** or the **residue** of the division.

Theorem: If $m|a$ and $m|b$ then $m|\alpha a + \beta b$ for any integers α, β.
Proof: $a = rm$; $b = sm$ for some r, s. Therefore, $\alpha a + \beta b = \alpha rm + \beta sm = m(\alpha r + \beta s)$, i.e., m divides this number. QED

Division (cont.)

If $n|(a-b)$, i.e., a and b have the same residues modulo n: $(a \bmod n) = (b \bmod n)$, we write $\boldsymbol{a \equiv b \pmod{n}}$ and say that a is **congruent** to b modulo n.
The integers can be divided into n equivalence classes according to their residue modulo n:

$$[a]_n = \{a + kn : k \in \mathbb{Z}\}$$

$$Z_n = \{[a]_n : 0 \leq a \leq n-1\}$$

or briefly

$$Z_n = \{0, 1, \ldots, n-1\}$$

Greatest Common Divisor

Let a and b be integers.

1. **gcd(a,b)** (the **greatest common divisor** of a and b) is

$$\gcd(a,b) \stackrel{\Delta}{=} \max(d : d|a \text{ and } d|b)$$

   (for $a \neq 0$ or $b \neq 0$).

   Note: This definition satisfies $\gcd(0,1) = 1$.

2. **lcm(a,b)** (the **least common multiplier** of a and b) is

$$\operatorname{lcm}(a,b) \stackrel{\Delta}{=} \min(d > 0 : a|d \text{ and } b|d)$$

   (for $a \neq 0$ and $b \neq 0$).

3. a and b are **coprimes** (or **relatively prime**) iff $\gcd(a,b) = 1$.

Greatest Common Divisor (cont.)

Theorem: Let a, b be integers, not both zero, and let d be the smallest positive element of $S = \{ax + by : x, y \in I\!N\}$. Then, $\gcd(a, b) = d$.

Proof: S contains a positive integer because $|a| \in S$.

By definition, there exist x, y such that $d = ax + by$. $d \leq |a|$, thus there exist q, r such that

$$a = qd + r, \qquad 0 \leq r < d.$$

Thus,

$$r = a - qd = a - q(ax + by) = a(1 - qx) + b(-qy) \in S.$$

$r < d$ implies $r = 0$, thus $d|a$.

By the same arguments we get $d|b$.

$d|a$ and $d|b$, thus $d \leq \gcd(a, b)$.

On the other hand $\gcd(a, b)|a$ and $\gcd(a, b)|b$, and thus $\gcd(a, b)$ divides any linear combination of a, b, i.e., $\gcd(a, b)$ divides all elements in S, including d, and thus $\gcd(a, b) \leq d$. We conclude that $d = \gcd(a, b)$. QED

Greatest Common Divisor (cont.)

Corollary: For any a, b, and d, if $d|a$ and $d|b$ then $d|\gcd(a, b)$.
Proof: $\gcd(a, b)$ is a linear combination of a and b.

Lemma: For $m \neq 0$

$$\gcd(ma, mb) = |m| \gcd(a, b).$$

Proof: If $m \neq 0$ (WLG $m > 0$) then $\gcd(ma, mb)$ is the smallest positive element in the set $\{amx+bmy\}$, which is m times the smallest positive element in the set $\{ax + by\}$.

Greatest Common Divisor (cont.)

Corollary: a and b are coprimes iff

$$\exists x, y \text{ such that } xa + yb = 1.$$

Proof:
($\Leftarrow$) Let $d = gcd(a, b)$, and $xa + yb = 1$. $d|a$ and $d|b$ and therefore, $d|1$, and thus $d = 1$.
($\Rightarrow$) a and b are coprimes, i.e., $\mathrm{gcd}(a, b) = 1$. Using the previous theorem, 1 is the smallest positive integer in $S = \{ax + by : x, y \in I\!N\}$, i.e., $\exists x, y$ such that $ax + by = 1$. QED

The Fundamental Theorem of Arithmetic

The fundamental theorem of arithmetic: If $c|ab$ and $\gcd(b, c) = 1$ then $c|a$.

Proof: We know that $c|ab$. Clearly, $c|ac$.

Thus,

$$c|\gcd(ab, ac) = a \cdot \gcd(b, c) = a \cdot 1 = a.$$

QED

Prime Numbers and Unique Factorization

Definition: An integer $p \geq 2$ is called **prime** if it is divisible only by 1 and itself.

Theorem: Unique Factorization: Every positive number can be represented as a product of primes in a unique way, up to a permutation of the order of primes.

Prime Numbers and Unique Factorization (cont.)

Proof: Every number can be represented as a product of primes, since if one element is not a prime, it can be further factored into smaller primes.
Assume that some number can be represented in two distinct ways as products of primes:

$$p_1 p_2 p_3 \cdots p_s = q_1 q_2 q_3 \cdots q_r$$

where all the factors are prime, and no p_i is equal to some q_j (otherwise discard both from the product).
Then,

$$p_1 | q_1 q_2 q_3 \cdots q_r.$$

But $\gcd(p_1, q_1) = 1$ and thus

$$p_1 | q_2 q_3 \cdots q_r.$$

Similarly we continue till

$$p_1 | q_r.$$

Contradiction. QED

Euclid's Algorithm

Let a and b be two positive integers, $a > b > 0$. Then the following algorithm computes $\gcd(a, b)$:

$r_{-1} = a$
$r_0 = b$
for i from 1 until $r_i = 0$
    $\exists q_i, r_i : r_{i-2} = q_i r_{i-1} + r_i$ and $0 \le r_i < r_{i-1}$
k=i-1

Example: $a = 53$ and $b = 39$.

$$\begin{aligned} 53 &= 1 \cdot 39 + 14 \\ 39 &= 2 \cdot 14 + 11 \\ 14 &= 1 \cdot 11 + 3 \\ 11 &= 3 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

Thus, $\gcd(53, 39) = 1$.

Extended Form of Euclid's Algorithm

Example (cont.): $a = 53$ and $b = 39$.

$$\begin{array}{rlcrl}
53= & 1\cdot 39+14 & \Rightarrow & 14= & 53-39 \\
39= & 2\cdot 14+11 & \Rightarrow & 11= & 39-2\cdot 14=-2\cdot 53+3\cdot 39 \\
14= & 1\cdot 11+3 & \Rightarrow & 3= & 14-1\cdot 11=3\cdot 53-4\cdot 39 \\
11= & 3\cdot 3+2 & \Rightarrow & 2= & 11-3\cdot 3=-11\cdot 53+15\cdot 39 \\
3= & 1\cdot 2+1 & \Rightarrow & 1= & 3-1\cdot 2=14\cdot 53-19\cdot 39 \\
2= & 2\cdot 1+0 & & &
\end{array}$$

Therefore, $14 \cdot 53 - 19 \cdot 39 = 1$.
We will use this algorithm later as a modular inversion algorithm, in this case we get that $(-19) \cdot 39 \equiv 34 \cdot 39 \equiv 1 \pmod{53}$.
Note that every r_i is written as a linear combination of r_{i-1} and r_{i-2}, and ultimately, r_i is written as a linear combination of a and b.

Proof of Euclid's Algorithm

Claim: The algorithm stops after at most $O(\log a)$ steps.
Proof: It suffices to show that in each step $r_i < r_{i-2}/2$:
For $i = 1$: $r_1 < b < a$ and thus in $a = q_1 b + r_1$, $q_1 \geq 1$. Therefore, $a \geq 1b + r_1 > r_1 + r_1$, and thus $a/2 > r_1$.
For $i > 1$: $r_i < r_{i-1} < r_{i-2}$ and thus $r_{i-2} = q_i r_{i-1} + r_i$, $q_i \geq 1$. Therefore, $r_{i-2} \geq 1r_{i-1} + r_i > r_i + r_i$, and thus $r_{i-2}/2 > r_i$.
After at most $2 \log a$ steps, r_i reduces to zero. QED

Proof of Euclid's Algorithm (cont.)

Claim: $r_k = \gcd(a, b)$.

Proof:

$r_k \mid \gcd(a, b)$: $r_k \mid r_{k-1}$ because of the stop condition. $r_k \mid r_k$ and $r_k \mid r_{k-1}$ and therefore r_k divides any linear combination of r_{k-1} and r_k, including r_{k-2}. Since $r_k \mid r_{k-1}$ and $r_k \mid r_{k-2}$, it follows that $r_k \mid r_{k-3}$. Continuing this way, it follows that $r_k \mid a$ and that $r_k \mid b$, thus $r_k \mid \gcd(a, b)$.

$\gcd(a, b) \mid r_k$: r_k is a linear combination of a and b; $\gcd(a, b) \mid a$ and $\gcd(a, b) \mid b$, therefore, $\gcd(a, b) \mid r_k$.

We conclude that $r_k = \gcd(a, b)$. QED

Groups

A **group** $(S, \oplus)$ is a set S with a binary operation $\oplus$ defined on S for which the following properties hold:

1. **Closure**: $a \oplus b \in S$ For all $a, b \in S$.
2. **Identity**: There is an element $e \in S$ such that $e \oplus a = a \oplus e = a$ for all $a \in S$.
3. **Associativity**: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in S$.
4. **Inverses**: For each $a \in S$ there exists an unique element $b \in S$ such that $a \oplus b = b \oplus a = e$.

If a group $(S, \oplus)$ satisfies the **commutative law** $a \oplus b = b \oplus a$ for all $a, b \in S$ then it is called an **Abelian group**.
Definition: The **order** of a group, denoted by $|S|$, is the number of elements in S. If a group satisfies $|S| < \infty$ then it is called a **finite group**.
Lemma: $(Z_n, +_n)$ is a finite Abelian **additive group** modulo n.

Groups (cont.)

Basic Properties:

Let:

$$a^k = \bigoplus_{i=1}^{k} a = \underbrace{a \oplus a \oplus \ldots \oplus a}_{k}.$$

$$a^0 = e$$

1. The identity element e in the group is unique.
2. Every element a has a **single** inverse, denoted by a^{-1}. We define $a^{-k} = \bigoplus_{i=1}^{k} a^{-1}$.
3. $a^m \oplus a^n = a^{m+n}$.
4. $(a^m)^n = a^{nm}$.

Groups (cont.)

Definition: The **order** of a in a group S is the least $t > 0$ such that $a^t = e$, and it is denoted by $\mathrm{order}(a, S)$.

For example, in the group $(Z_3, +_3)$, the order of 2 is 3 since $2 + 2 \equiv 4 \equiv 1$, $2 + 2 + 2 \equiv 6 \equiv 0$ (and 0 is the identity in Z_3).

Subgroups

Definition: If $(S,\oplus)$ is a group, $S' \subseteq S$, and $(S',\oplus)$ is also a group, then $(S',\oplus)$ is called a **subgroup** of $(S,\oplus)$.

Theorem: If $(S,\oplus)$ is a finite group and S' is any subset of S such that $a \oplus b \in S'$ for all $a, b \in S'$, then $(S',\oplus)$ is a subgroup of $(S,\oplus)$.

Example: $(\{0,2,4,6\},+_8)$ is a subgroup of $(Z_8,+_8)$, since it is closed under the operation $+_8$.

Lagrange's theorem: If $(S,\oplus)$ is a finite group and $(S',\oplus)$ is a subgroup of $(S,\oplus)$ then $|S'|$ is a divisor of $|S|$.

Subgroups (cont.)

Let a be an element of a group S, denote by $(\langle a\rangle, \oplus)$ the set:

$$\langle a\rangle = \{a^k : \text{order}(a, S) \geq k \geq 1\}$$

Theorem: $\langle a\rangle$ contains $\text{order}(a, S)$ distinct elements.
Proof: Assume by contradiction that there exists $1 \leq i < j \leq \text{order}(a, S)$, such that $a^i = a^j$. Therefore, $e = a^{j-i}$ in contradiction to fact that $\text{order}(a, S) > j - i > 0$. QED
Lemma: $\langle a\rangle$ is a subgroup of S with respect to $\oplus$.
We say that a **generates** the subgroup $\langle a\rangle$ or that a is a **generator** of $\langle a\rangle$.
Clearly, the order of $\langle a\rangle$ equals the order of a in the group. $\langle a\rangle$ is also called a **cyclic** group.
Example: $\{0, 2, 4, 6\} \subset Z_8$ can be generated by 2 or 6.
Note that a cyclic group is always Abelian.

Subgroups (cont.)

Corollary: The order of an element divides the order of group.
Corollary: Any group of prime order must be cyclic.
Corollary: Let S be a finite group, and $a \in S$, then $a^{|S|} = e$.
Theorem: Let a be an element in a group S, such that $a^s = e$, then $\text{order}(a, S)|s$.
Proof: Using the division theorem, $s = q \cdot \text{order}(a, S) + r$, where $0 \leq r < \text{order}(a, S)$. Therefore,

$$e = a^s = a^{q \cdot \text{order}(a,S)+r} = (a^{\text{order}(a,S)})^q \oplus a^r = a^r.$$

Due to the minimality of $\text{order}(a, S)$, we conclude that $r = 0$. QED

Fields

Definition: A **Field** $(S, \oplus, \odot)$ is a set S with two binary operations $\oplus$ and $\odot$ defined on S and with two special elements denoted by $0, 1$ for which the following properties hold:

1. $(S, \oplus)$ is an Abelian group (0 is the identity with regards to $\oplus$).
2. $(S \setminus \{0\}, \odot)$ is an Abelian group (1 is the identity with regards to $\odot$).
3. **Distributivity**: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

Corollary: $\forall a \in S, a \odot 0 = 0$.
Proof: $a \odot 0 = a \odot (0 \oplus 0) = a \odot 0 \oplus a \odot 0$, thus, $a \odot 0 = 0$.
Examples: $(Q, +, \cdot)$, $(Z_p, +_p, \cdot_p)$ where p is a prime.

Inverses

Lemma: Let p be a prime. Then,

$$ab \equiv 0 \pmod{p}$$

iff

$$a \equiv 0 \pmod{p} \quad \text{or} \quad b \equiv 0 \pmod{p}.$$

Proof:
($\Leftarrow$) From $p|a$ or $p|b$ it follows that $p|ab$.
($\Rightarrow$) $p|ab$. If $p|a$ we are done. Otherwise, $p \not| a$.
Since p a prime it follows that $\gcd(a, p) = 1$. Therefore, $p|b$ (by the fundamental theorem of arithmetic). QED

Inverses (cont.)

Definition: Let a be a number. If there exists b such that $ab \equiv 1 \pmod{m}$, then we call b the **inverse** of a modulo m, and write $b \stackrel{\Delta}{=} a^{-1} \pmod{m}$.

Theorem: If $\gcd(a,m) = 1$ then there exists some b such that $ab \equiv 1 \pmod{m}$.

Proof: There exist x, y such that

$$xa + ym = 1.$$

Thus,

$$xa \equiv 1 \pmod{m}.$$

QED

Conclusion: a has an inverse modulo m iff $\gcd(a,m) = 1$. The inverse can be computed by Euclid's algorithm.

Z_n^*

Definition: Z_n^* is the set of all the invertible integers modulo n:

$$Z_n^* = \{i \in Z_n |\, \gcd(i, n) = 1\}.$$

Theorem: For any positive n, Z_n^* is an Abelian **multiplicative** group under multiplication modulo n.
Proof: Exercise.
Z_n^* is also called an Euler group.
Example: For a prime p, $Z_p^* = \{1, 2, \ldots, p-1\}$.

Z_n^* (cont.)

Examples:

$$
\begin{array}{ll}
Z_2 = \{0,1\} & Z_2^* = \{1\} \\
Z_3 = \{0,1,2\} & Z_3^* = \{1,2\} \\
Z_4 = \{0,1,2,3\} & Z_4^* = \{1,3\} \\
Z_5 = \{0,1,2,3,4\} & Z_5^* = \{1,2,3,4\}
\end{array}
$$

$$
Z_1 = \{0\} \qquad Z_1^* = \{0\} \quad \text{!!!!!}
$$

Euler's Function

Definition: Euler's function $\varphi(n)$ represents the number of elements in Z_n^*:

$$\varphi(n) \stackrel{\Delta}{=} |Z_n^*| = |\{i \in Z_n |\, \gcd(i,n) = 1\}|$$

$\varphi(n)$ is the number of numbers in $\{0, \ldots, n-1\}$ that are coprime to n.

Note that by this definition $\varphi(1) \stackrel{\Delta}{=} 1$ (since $Z_1^* = \{0\}$, which is because $\gcd(0,1) = 1$).

Euler's Function (cont.)

Theorem: Let $n = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$ be the unique factorization of n to distinct primes. Then,

$$\varphi(n) = \prod (p_i^{e_i - 1}(p_i - 1)) = n \prod (1 - \frac{1}{p_i}).$$

Proof: Exercise.
Note: If the factorization of n is not known, $\varphi(n)$ is not known as well.
Conclusions: For prime numbers $p \neq q$, and any integers a and b

1. $\varphi(p) = p - 1$.
2. $\varphi(p^e) = (p-1)p^{e-1} = p^e - p^{e-1}$.
3. $\varphi(pq) = (p-1)(q-1)$.
4. If $\gcd(a, b) = 1$ then $\varphi(ab) = \varphi(a)\varphi(b)$.

Euler's Function (cont.)

Theorem:

$$\sum_{d|n} \varphi(d) = n.$$

Proof: In this proof, we count the numbers $1, \ldots, n$ in a different order. We divide the numbers into distinct groups according to their gcd d' with n, thus the total number of elements in the groups is n.

It remains to see what is the number of numbers out of $1, \ldots, n$ whose gcd with n is d'.

Clearly, if $d' \not| n$, the number is zero.

Otherwise, let $d'|n$ and $1 \leq a \leq n$ be a number such that $\gcd(a, n) = d'$. Therefore, $a = kd'$, for some $k \in \{1, \ldots, n/d'\}$. Substitute a with kd', thus $\gcd(kd', n) = d'$, i.e., $\gcd(k, n/d') = 1$.

Euler's Function (cont.)

It remains to see for how many k's, $1 \leq k \leq n/d'$, it holds that

$$\gcd(k, n/d') = 1.$$

But this is the definition of Euler's function, thus there are $\varphi(n/d')$ such k's.
Since we count each a exactly once

$$\sum_{d'|n} \varphi(n/d') = n.$$

If $d'|n$ then also $d = \frac{n}{d'}$ divides n, and thus we can substitute n/d' with d and get

$$\sum_{d|n} \varphi(d) = n.$$

QED

Euler's Theorem

Theorem: For any a and m, if $\gcd(a, m) = 1$ then

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Proof: a is an element in the Euler group Z_m^*. Therefore, as a corollary from Lagrange Theorem, $a^{|Z_m^*|} = a^{\varphi(m)} = 1 \pmod{m}$. QED

Fermat's Little Theorem

Fermat's little theorem: Let p be a prime number. Then, any integer a satisfies

$$a^p \equiv a \pmod{p}.$$

Proof: If $p|a$ the theorem is trivial, as $a \equiv 0 \pmod{p}$. Otherwise p and a are coprimes, and thus by Euler's theorem

$$a^{p-1} \equiv 1 \pmod{p}$$

and

$$a^p \equiv a \pmod{p}.$$

QED

Properties of Elements in the Group Z_m^*

Definition: For a, m such that $\gcd(a, m) = 1$, let h be the smallest integer ($h > 0$) satisfying

$$a^h \equiv 1 \pmod{m}.$$

(Such an integer exists by Euler's theorem: $a^{\varphi(m)} \equiv 1 \pmod{m}$). We call h the **order of a modulo m**, and write $h = \textbf{order}(a, Z_m^*)$.
Obviously, it is equivalent to the order of a in the Euler group Z_m^*.

Properties of Elements in the Group Z_m^* (cont.)

Conclusions: For a, m such that $\gcd(a, m) = 1$

1. If $a^s \equiv 1 \pmod{m}$, then $\mathrm{order}(a, Z_m^*) | s$.
2. $\mathrm{order}(a, Z_m^*) | \varphi(m)$
3. If m is a prime, then $\mathrm{order}(a, Z_m^*) | m - 1$.
4. The numbers
$$1, a^1, a^2, a^3, \ldots, a^{\mathrm{order}(a, Z_m^*)-1}$$
are all distinct modulo m.

Proof: Follows from the properties of groups.
QED

Modular Exponentiation

Given a prime q and $a \in Z_q^*$ we want to calculate $a^x \bmod q$.
Denote x in binary representation as

$$x = x_{n-1}x_{n-2}\ldots x_1 x_0,$$

where $x = \sum_{i=0}^{n-1} x_i 2^i$.
Therefore, $a^x \bmod q$ can be written as:

$$a^x = a^{2^{(n-1)}x_{n-1}} a^{2^{(n-2)}x_{n-2}} \cdots a^{2x_1} a^{x_0}$$

An Algorithm for Modular Exponentiation

$$a^x = a^{2^{(n-1)}x_{n-1}} a^{2^{(n-2)}x_{n-2}} \cdots a^{2x_1} a^{x_0}$$

Algorithm:

$r \leftarrow 1$
for $i \leftarrow n-1$ down to 0 do
$\quad r \leftarrow r^2 a^{x_i} \bmod q \qquad (a^{x_i}$ is either 1 or $a)$

At the end

$$r = \prod_{i=0}^{n-1} a^{x_i 2^i} = a^{\left(\sum_{i=0}^{n-1} x_i 2^i\right)} = a^x \pmod{q}.$$

Complexity: $O(\log x)$ modular multiplications. For a random x this complexity is $O(\log q)$.

An Algorithm for Modular Exponentiation (cont.)

An important note:

$$(xy) \bmod q = ((x \bmod q)(y \bmod q)) \bmod q,$$

i.e., the modular reduction can be performed every multiplication, or only at the end, and the results are the same.
The proof is given as an exercise.

The Chinese Remainder Theorem

Problem 1: Let $n = pq$ and let $x \in Z_n$. Compute $x \bmod p$ and $x \bmod q$.
Both are easy to compute, given p and q.

Problem 2: Let $n = pq$, let $x \in Z_p$ and let $y \in Z_q$. Compute $u \in Z_n$ such that

$$\begin{aligned} u &\equiv x \pmod p \\ u &\equiv y \pmod q. \end{aligned}$$

The Chinese Remainder Theorem (cont.)

Generalization: Given moduli m_1, m_2, ..., m_k and values y_1, y_2, ..., y_k. Compute u such that for any $i \in \{1, \ldots, k\}$

$$u \equiv y_i \pmod{m_i}.$$

We can assume (without loss of generality) that all the m_i's are coprimes in pairs ($\forall_{i \neq j} \gcd(m_i, m_j) = 1$). (If they are not coprimes in pairs, either they can be reduced to an equivalent set in which they are coprimes in pairs, or else the system leads to a contradiction, such as $u \equiv 1 \pmod{3}$ and $u \equiv 2 \pmod{6}$).

Example: Given the moduli $m_1 = 11$ and $m_2 = 13$ find a number $u \pmod{11 \cdot 13}$ such that $u \equiv 7 \pmod{11}$ and $u \equiv 4 \pmod{13}$.

Answer: $u \equiv 95 \pmod{11 \cdot 13}$. Check: $95 = 11 \cdot 8 + 7$, $95 = 13 \cdot 7 + 4$.

The Chinese Remainder Theorem (cont.)

The Chinese remainder theorem: Let m_1, m_2, ..., m_k be coprimes in pairs and let y_1, y_2, ..., y_k. Then, there is an **unique solution** u modulo $m = \prod m_i = m_1 m_2 \cdots m_k$ of the equations:

$$\begin{aligned} u &\equiv y_1 \pmod{m_1} \\ u &\equiv y_2 \pmod{m_2} \\ &\vdots \\ u &\equiv y_k \pmod{m_k}, \end{aligned}$$

and it can be **efficiently computed**.

The Chinese Remainder Theorem (cont.)

Example: Let

$$u \equiv 7 \pmod{11} \qquad u \equiv 4 \pmod{13}$$

then compute

$$u \equiv ? \pmod{11 \cdot 13}.$$

Assume we found two numbers a and b such that

$$a \equiv 1 \pmod{11} \qquad a \equiv 0 \pmod{13}$$

and

$$b \equiv 0 \pmod{11} \qquad b \equiv 1 \pmod{13}$$

Then,

$$u \equiv 7a + 4b \pmod{11 \cdot 13}.$$

The Chinese Remainder Theorem (cont.)

We remain with the problem of finding a and b. Notice that a is divisible by 13, and $a \equiv 1 \pmod{11}$.

Denote the inverse of 13 modulo 11 by $c \equiv 13^{-1} \pmod{11}$. Then,

$$\begin{aligned} 13c &\equiv 1 \pmod{11} \\ 13c &\equiv 0 \pmod{13} \end{aligned}$$

We conclude that

$$a \equiv 13c \equiv 13(13^{-1} \pmod{11}) \pmod{11 \cdot 13}$$

and similarly

$$b \equiv 11(11^{-1} \pmod{13}) \pmod{11 \cdot 13}$$

Thus,

$$u \equiv 7 \cdot 13 \cdot 6 + 4 \cdot 11 \cdot 6 \equiv 810 \equiv 95 \pmod{11 \cdot 13}$$

The Chinese Remainder Theorem (cont.)

Proof: m/m_i and m_i are coprimes, thus m/m_i has an inverse modulo m_i. Denote

$$l_i \equiv (m/m_i)^{-1} \pmod{m_i}$$

and

$$b_i = l_i(m/m_i).$$

$$\begin{aligned} b_i &\equiv 1 \pmod{m_i} \\ b_i &\equiv 0 \pmod{m_j}, \qquad \forall j \neq i \quad (\text{since } \; m_j | (m/m_i)). \end{aligned}$$

The solution is

$$\begin{aligned} u &\equiv y_1 b_1 + y_2 b_2 + \cdots + y_k b_k \\ &\equiv \sum_{i=1}^{m} y_i b_i \pmod{m}. \end{aligned}$$

The Chinese Remainder Theorem (cont.)

We still have to show that the solution is unique modulo m. By contradiction, we assume that there are two distinct solutions u_1 and u_2, $u_1 \not\equiv u_2 \pmod{m}$. But any modulo m_i satisfy $u_1 - u_2 \equiv 0 \pmod{m_i}$, and thus

$$m_i | u_1 - u_2.$$

Since m_i are pairwise coprimes we conclude that

$$m = \prod m_i | u_1 - u_2$$

which means that

$$u_1 - u_2 \equiv 0 \pmod{m}.$$

Contradiction. QED

$Z^*_{ab} \equiv Z^*_a \times Z^*_b$

Consider the homomorphism $\Psi : Z^*_{ab} \rightarrow Z^*_a \times Z^*_b$,
$\Psi(u) = (\alpha = u \bmod a, \beta = u \bmod b)$.

Lemma: $u \in Z^*_{ab}$ iff $\alpha \in Z^*_a$ and $\beta \in Z^*_b$, i.e.,
$\gcd(ab, u) = 1$ iff $\gcd(a, u) = 1$ and $\gcd(b, u) = 1$.

Proof:

($\Rightarrow$) Trivial ($k_1 ab + k_2 u = 1$ for some k_1 and k_2).

($\Leftarrow$) By the assumptions there exist some k_1, k_2, k_3, k_4 such that

$$k_1 a + k_2 u = 1 \text{ and } k_3 b + k_4 u = 1.$$

Thus,

$$k_1 a(k_3 b + k_4 u) + k_2 u = 1$$

from which we get

$$k_1 k_3 ab + (k_1 k_4 a + k_2)u = 1.$$

QED

$Z^*_{ab} \equiv Z^*_a \times Z^*_b$ (cont.)

Lemma: Ψ is onto.

Proof: Choose any $\alpha \in Z^*_a$ and any $\beta \in Z^*_b$, we can reconstruct u, using the Chinese remainder theorem, and $u \in Z^*_{ab}$ from previous lemma.

Lemma: Ψ is one to one.

Proof: Assume to the contrary that for $\alpha \in Z^*_a$ and $\beta \in Z^*_b$ there are $u_1 \not\equiv u_2 \pmod{ab}$. This is a contradiction to the uniqueness of the solution of the Chinese remainder theorem.

QED

We conclude from the Chinese remainder theorem and these two Lemmas that Z^*_{ab} is 1-1 related to $Z^*_a \times Z^*_b$.

For every $\alpha \in Z^*_a$ and $\beta \in Z^*_b$ there exists a unique $u \in Z^*_{ab}$ such that $u \equiv \alpha \pmod{a}$ and $u \equiv \beta \pmod{b}$, and vise versa.

Note: This can be used to construct an alternative proof for $\varphi(pq) = \varphi(p)\varphi(q)$, where $\gcd(p, q) = 1$.

Lagrange's Theorem

Theorem: A polynomial of degree $n > 0$

$$f(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n$$

has at most n distinct roots modulo a prime p.

Proof: It is trivial for $n = 1$.

By induction:

Assume that any polynomial of degree $n-1$ has at most $n-1$ roots. Let a be a root of $f(x)$, i.e., $f(a) \equiv 0 \pmod{p}$.

We can write

$$f(x) = (x-a)f_1(x) + r \pmod{p}$$

for some polynomial $f_1(x)$ and constant r (this is a division of $f(x)$ by $(x-a)$).

Since $f(a) \equiv 0 \pmod{p}$ then $r \equiv 0 \pmod{p}$ and we get

$$f(x) = (x-a)f_1(x) \pmod{p}.$$

Thus, any root $b \neq a$ of $f(x)$ is also a root of $f_1(x)$:

$$0 \equiv f(b) \equiv (b-a)f_1(b) \pmod{p}$$

Lagrange's Theorem (cont.)

which causes

$$f_1(b) \equiv 0 \pmod{p}.$$

f_1 is of degree $n-1$, and thus has at most $n-1$ roots. Together with a, f has at most n roots. QED

Note: Lagrange's Theorem does not hold for composites, for example:

$$x^2 - 4 \equiv 0 \pmod{35}$$

has 4 roots: $2, 12, 23$ and 33.

Generators

Definition: a is called a **generator** of Z_n^* if $\text{order}(a, Z_n^*) = \varphi(n)$.
Not all groups posses generators. If Z_n^* possesses a generator g, then Z_n^* is **cyclic**.
If g is a generator of Z_n^* and a is any element of Z_n^* then there exists a z such that $g^z \equiv a \pmod{n}$. This z is called the **discrete logarithm** or **index** of a modulo n to the base g. We denote this value as $\text{ind}_{n,g}(a)$ or $\text{DLOG}_{n,g}(a)$.

The Number of Generators

Theorem: Let h be the order of a modulo m. Let s be an integer such that $\gcd(h, s) = 1$, then the order of a^s modulo m is also h.
Proof: Denote the order of a by h and the order of a^s by h'.

$$(a^s)^h \equiv (a^h)^s \equiv 1 \pmod{m}.$$

Thus, $h'|h$.
On the other hand,

$$a^{sh'} \equiv (a^s)^{h'} \equiv 1 \pmod{m}$$

and thus $h|sh'$. Since $\gcd(h, s) = 1$ then $h|h'$.
QED

The Number of Generators (cont.)

Theorem: Let p be a prime and $d|p-1$. The number of integers in Z_p^* of order d is $\varphi(d)$.

Proof: Denote the number of integers in Z_p^* which are of order d by $\psi(d)$. We should prove that $\psi(d) = \varphi(d)$.

Assume that $\psi(d) \neq 0$, and let $a \in Z_p^*$ have an order d ($a^d \equiv 1 \pmod{p}$).

The equation $x^d \equiv 1 \pmod{p}$ has the following solutions

$$1 \equiv a^d, a^1, a^2, a^3, \ldots, a^{d-1},$$

all of which are distinct.

We know that $x \equiv a^i \pmod{p}$ has an order of d iff $\gcd(i, d) = 1$, and thus the number of solutions with order d is $\psi(d) = \varphi(d)$.

The Number of Generators (cont.)

We should show that the equality holds even if $\psi(d) = 0$. Each of the integers in $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ has some order $d|p-1$. Thus, the sum of $\psi(d)$ for all the orders $d|p-1$ equals $|Z_p^*|$:

$$\sum_{d|p-1} \psi(d) = p - 1.$$

As we know that $\sum_{d|p-1} \varphi(d) = p - 1$, it follows that:

$$\begin{aligned}
0 &= \sum_{d|p-1} (\varphi(d) - \psi(d)) = \\
&= \sum_{d|p-1, \psi(d)=0} (\varphi(d) - \psi(d)) + \sum_{d|p-1, \psi(d)\neq 0} (\varphi(d) - \psi(d)) = \\
&= \sum_{d|p-1, \psi(d)=0} \varphi(d) + \sum_{d|p-1, \psi(d)\neq 0} 0 = \sum_{d|p-1, \psi(d)=0} \varphi(d)
\end{aligned}$$

The Number of Generators (cont.)

Since $\varphi(d) \geq 0$, then $\psi(d) = 0 \Rightarrow \varphi(d) = 0$. We conclude that for any d:

$$\psi(d) = \varphi(d).$$

QED

The Number of Generators (cont.)

Conclusion: Let p be a prime. There are $\varphi(p-1)$ elements in Z_p^* of order $p-1$ (i.e., all of them are generators).
Therefore, Z_p^* is cyclic.
Theorem: The values of $n>1$ for which Z_n^* is cyclic are $2, 4, p^e$ and $2p^e$ for all odd primes p and all positive integers e.
Proof: Exercise.

Wilson's Theorem

Wilson's theorem: Let p be a prime.

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (p-1) \equiv -1 \pmod{p}.$$

Proof: Clearly it holds for $p = 2$. It suffices thus to prove it for $p \geq 3$.
Let g be a generator of Z_p^*. Then,

$$Z_p^* = \{1, g, g^2, g^3, \ldots, g^{p-2}\}$$

and thus

$$\begin{aligned} 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (p-1) &\equiv 1 \cdot g \cdot g^2 \cdot g^3 \cdot \ldots \cdot g^{p-2} \\ &\equiv g^{(p-2)(p-1)/2} \pmod{p}. \end{aligned}$$

Wilson's Theorem (cont.)

If $g^{(p-1)/2} \equiv -1 \pmod{p}$, then it follows that

$$\begin{aligned} 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (p-1) &\equiv g^{(p-2)(p-1)/2} \pmod{p} \\ &\equiv (-1)^{p-2} \equiv -1 \pmod{p}. \end{aligned}$$

It remains to show that $g^{(p-1)/2} \equiv -1 \pmod{p}$. From Euler theorem it follows that

$$g^{p-1} \equiv 1 \pmod{p}.$$

Thus,

$$0 \equiv g^{p-1} - 1 \equiv (g^{(p-1)/2} + 1)(g^{(p-1)/2} - 1) \pmod{p}.$$

$g^{(p-1)/2} \not\equiv 1 \pmod{p}$ since order$(g, Z_p^*) = p - 1$ (and p is odd), and thus it must be that $g^{(p-1)/2} \equiv -1 \pmod{p}$.

QED