

Near-Optimal Compression for the Planar Graph Metric

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Abstract

The Planar Graph Metric Compression Problem is to compactly encode the distances among k nodes in a planar graph of size n . Two naïve solutions are to store the graph using $O(n)$ bits, or to explicitly store the distance matrix with $O(k^2 \log n)$ bits. The only lower bounds are from the seminal work of Gavaille, Peleg, Prennes, and Raz [SODA'01], who rule out compressions into a polynomially smaller number of bits, for *weighted* planar graphs, but leave a large gap for unweighted planar graphs. For example, when $k = \sqrt{n}$, the upper bound is $O(n)$ and their constructions imply an $\Omega(n^{3/4})$ lower bound. This gap is directly related to other major open questions in labeling schemes, dynamic algorithms, and compact routing.

Our main result is a new compression of the planar graph metric into $\tilde{O}(\min(k^2, \sqrt{k \cdot n}))$ bits, which is optimal up to log factors. Our data structure circumvents an $\Omega(k^2)$ lower bound of Krauthgamer, Nguyen, and Zondiner [SIDMA'14] for compression using minors, and the lower bound of Gavaille et al. for compression of weighted planar graphs. This is an unexpected and decisive proof that weights can make planar graphs inherently more complex. Moreover, we design a new *Subset Distance Oracle* for planar graphs with $\tilde{O}(\sqrt{k \cdot n})$ space, and $\tilde{O}(n^{3/4})$ query time.

Our work carries strong messages to related fields. In particular, the famous $O(n^{1/2})$ vs. $\Omega(n^{1/3})$ gap for distance labeling schemes in planar graphs *cannot* be resolved with the current lower bound techniques. On the positive side, we introduce the powerful tool of unit-monge to planar graph algorithms.

1 Introduction

The shortest path metric of planar graphs is one of the most popular and well-studied metrics in Computer Science. Countless papers, surveys, and textbooks address the computational challenges that arise when dealing with it. In this paper, we address a core problem about this metric that has remained poorly understood. We ask: *How compressible is it?* That is, how many bits do we need, information theoretically, in order to describe a set of distances in a planar graph?

As we discuss shortly, a better understanding of this core question is crucial to making progress on big open problems in other well-studied subjects such as Sparsification, Labeling Schemes, and Dynamic Data Structures. But first, let us define our problem more formally. In the *Metric Compression* problem, we are given a set S of k points in some metric space with distance function d , such as the metric of distances in an n node planar graph, and the goal is to find an encoding \mathcal{C} that is as short as possible, yet still allows us to compute $d(v_i, v_j)$ for any two points $v_i, v_j \in S$.

DEFINITION 1. THE PLANAR GRAPH METRIC COMPRESSION PROBLEM. *Given a graph G from the family of unweighted, undirected planar graphs with n nodes, and a subset S of k distinguished nodes in G , compute a bit string \mathcal{C} that encodes the distances between all pairs of nodes in S . That is, there is a decoding function f (the same function for the entire graph family) that given the encoding \mathcal{C} and any two nodes $v_i, v_j \in S$ returns the v_i -to- v_j distance in G .*

There are two naïve ways to solve this problem. First, we can store all the distances explicitly as a $k \times k$ matrix in the encoding \mathcal{C} . The distance in a graph on n nodes is some number in $\{0, 1, \dots, n\}$, and so this matrix can be encoded using $O(k^2 \log n)$ bits. The second option, which is better whenever $k^2 > n$, is to let the encoding be the graph G itself. Naïvely, this is $O(n \log n)$ bits, and more sophisticated encodings give $O(n)$ [17, 25, 64, 76]. The naïve upper bound for our problem is therefore $\tilde{O}(\min\{k^2, n\})$.¹ Is this the best possible, or could there be a much better compression into $\tilde{O}(k \cdot n^\epsilon)$ or even $\tilde{O}(k)$ bits?

For context, let us look at other metrics. One example of a metric that admits an ultra-efficient compression into $\tilde{O}(k)$ bits is the metric of trees or bounded treewidth graphs [23, 38]. For most metrics of interest, however, the exact or lossless version of the compression problem is too difficult and no non-trivial upper bounds, beyond log-factor improvements, are possible. For example, in general (non-planar) graphs there is a simple $\Omega(k^2)$ lower bound: in any compression, each of

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¹We use the $\tilde{O}(\cdot)$ notation to hide polylogarithmic factors (polylogarithmic in n).

the $2^{\binom{k}{2}}$ possible graphs on k nodes must be encoded differently. Instead, it is popular to seek the optimal *lossy* compression from which the metric can be recovered *approximately*, e.g. up to a multiplicative $(1 + \varepsilon)$ error. Indeed, if we are willing to pay a $(1 + \varepsilon)$ error (for some fixed ε), then there are ingenious compressions of the planar graph metric into $\tilde{O}(k)$ bits [52, 56, 71]. But do we have to pay this error, or are planar graphs restricted enough to allow for non-trivial compression?

OPEN QUESTION 1. *Can we beat $\tilde{O}(\min\{k^2, n\})$ bits for planar graph metric compression?*

There are some lower bounds in our way. From the seminal work of Gavaille, Peleg, Pérennes, and Raz [38] we know that the metric of *weighted* planar graphs, where the edge weights are polynomially bounded, does not admit any non-trivial compression. The authors show that any Boolean $k \times k$ matrix can be “encoded” using the distances among a set of $2k$ nodes in a weighted planar graph on $n = O(k^2)$ nodes, where the edge weights are in $[k]$. Since we cannot compress an arbitrary $k \times k$ matrix into less than k^2 bits, we get a nearly-tight lower bound of $\Omega(\min\{k^2, n\})$ for *weighted* planar graphs. For unweighted planar graphs, Gavaille et al. simply subdivide the edges in their construction and the number of nodes in the encoding grows to $n = \Theta(k^3)$, which leads to a much weaker lower bound of $\Omega(\sqrt{k} \cdot n)$ (see Section 4 for more details). For example, when $k = \sqrt{n}$, the upper bound is $\tilde{O}(n)$ and the lower bound is $\Omega(n^{3/4})$. This subdivision of edges is rather naïve, and the overall lower bound construction does not seem to capture the full power of the planar graph metric. In fact, it can be simulated by a grid graph [3]. This naturally suggests the following intriguing challenge of finding a more clever encoding of matrices into planar graphs, which would lead to a negative resolution to Open Question 1.

CHALLENGE 1. *Can we encode an arbitrary $k \times k$ Boolean matrix M using the distances among a subset of $2k$ nodes $\{v_1, \dots, v_{2k}\}$ in an unweighted planar graph with $O(k^2)$ vertices, so that we can determine $M[i, j]$ by only looking at the distance between v_i and v_{k+j} in our graph?*

Before presenting our results, let us discuss the state of the art on questions that are closely related to ours, in which we are interested in data structures that are not only as succinct as possible, but also have other desirable features.

Sparsification. A natural way to compress a graph is by deleting or contracting some of its edges and nodes. Finding small subgraphs or minors that preserve or approximate the distances among a given

subset of k nodes have been studied for planar graphs [16, 22, 24, 32, 33, 43, 44, 47, 58, 59] and for general graphs, e.g. [27, 29, 68]. Such compressions are appealing algorithmically, since we can readily feed them into our usual graph algorithms, and recent research suggests that, in many settings, near-optimal compression bounds can be achieved using such sparsifiers (e.g. when compressing general graphs with additive error [1, 2]). A discouraging lower bound of Krauthgamer, Nguyen, and Zondiner [58] shows that even in the case of unweighted grid graphs, it is impossible to beat the naïve bound using a (possibly weighted) minor. Thus, a positive answer to Open Question 1 will have to involve a more complicated data structure.

Labeling Schemes. An appealing way to represent graphs is to assign a label ℓ_v to each node v , so that by looking at the labels of two nodes ℓ_s, ℓ_t we can infer certain properties such as the distance between them $d(s, t)$. Finding so-called *distance labeling schemes* in which the labels are as short as possible is a classical subject of study [38, 45, 50, 67]. Such labels are used for efficient algorithms both in theory [4, 72] and practice [30]. A famous open question is to close the rare *polynomial* gap in the bounds for planar graphs that has been embarrassingly open since the work of Gavaille et al. [38]: the upper bound is $O(n^{1/2})$ bits per label (due to [42] who shaved a log factor over [38]), and the lower bound is $\Omega(n^{1/3})$. The only known technique to prove polynomial lower bounds² is to argue that labeling schemes are one way to compress graphs, and then use facts about the limits of graph compression. For example, the lower bound for distance labeling of planar graphs [38] follows because labels of size $O(n^{1/3-\varepsilon})$ can be used to solve the metric compression problem using $O(k \cdot n^{1/3-\varepsilon})$ bits, which contradicts the lower bounds above. In fact, the tight lower bound for metric compression of *weighted* planar graphs leads to a tight lower bound for labeling schemes [3, 38]. Thus, to prove a tight lower bound of $\Omega(n^{1/2})$ for labeling schemes in unweighted planar graphs, the *only* approach we have with current techniques is to negatively resolve Open Question 1, e.g. by accomplishing Challenge 1.

Routing and Dynamic Algorithms. A compact routing scheme assigns names and tables to the nodes of a graph, so that each node s can find out the first edge on the shortest path (or some approximate path) to any target node t only using the name of t and the

²The only result that somewhat deviate from this technique are $1.008 \log n$ lower bound for nearest common ancestors in trees [12] and $1/8 \log^2 n$ lower bound for distance in trees [11, 38]. The gist of both of them is being able to argue about how much information can be shared by labels of two nodes. If the graph is not a tree, this seems very challenging.

local table stored at s . There is a vast literature on the topic, seeking the best possible tradeoff between sizes of the tables and the stretch in many different graph families (we refer the reader to Peleg’s book [66] and the extensive surveys [36, 37]). For planar graphs, Abraham, Gavoille, and Malkhi [5] write: “*Surprisingly, for stretch 1, the complexity of the size of the routing tables is not known.*” A simple upper bound is $\tilde{O}(n \cdot \sqrt{n})$ total table size, and an adaptation of the same Gavoille et al. construction gives a lower bound of $\Omega(n \cdot n^{1/3})$ [5]. It is likely that accomplishing Challenge 1 would resolve this gap as well. Yet another problem with similar state-of-the-art is the All Pairs Shortest Paths problem in *dynamic* planar graphs. Here, the goal is to have a data structure that supports efficient updates to the graph (edge additions or removals), and can answer shortest path queries efficiently. The breakthrough algorithm of Fakcharoenphol and Rao [35], and the later optimizations [39, 49, 51, 56], achieve $\tilde{O}(n^{2/3})$ time for updates and queries. The only framework for showing polynomial lower bound was recently proposed by Abboud and Dahlgaard [3] who proved a lower bound of $n^{1/3-o(1)}$ under the popular APSP Conjecture. Using their framework, accomplishing Challenge 1 directly leads to a higher lower bound of $n^{1/2-o(1)}$, as is known in the weighted case.

History suggests that weighted planar graph metrics might be harder to work with, but they are never *truly* harder. In so many cases, a new algorithm for the unweighted case is followed by an almost-as-good algorithm for the weighted case, a few years later. For example, a PTAS for the Travelling Salesman Problem in the unweighted planar metric was found in 1995 [46], and then for the weighted case in 1998 [14]. Perhaps it is only a matter of time until our lower bounds for the unweighted metric match the weighted.

1.1 Our Results Our first result is a new compression scheme for the planar graph metric, which achieves the information theoretically best possible bit complexity, up to log-factors. We give a *positive* resolution to Open Question 1, deem Challenge 1 to be infeasible, and show that unweighted planar graphs are inherently less complex than weighted ones; in fact, they admit a polynomially more efficient metric compression.

THEOREM 1.1. *Given an unweighted undirected planar graph on n nodes and a subset S of k nodes, we can return a binary encoding of length $O(\sqrt{k \cdot n} \cdot \log^3 n)$ from which all pairwise distances in S can be recovered exactly.*

This shrinks the gap in our understanding of the planar metric compression problem from polynomial

to polylogarithmic (removing this polylogarithmic gap remains an open question). For comparison, when $k = \sqrt{n}$, we show that $\Theta(n^{3/4})$ bits are necessary and sufficient, while in the weighted case the bound is $\Theta(n)$. Our encoding circumvents the lower bound of Krauthgamer et al. [58] for compressions using minors, and raises the question whether it can be matched via other forms of sparsification or *graphical* compressions.

It is unclear whether our new compression scheme will lead to improved upper bounds for labeling, routing, or dynamic algorithms. In Section 6, we discuss the difficulty in turning it into a labeling scheme. Still, it certainly shakes our beliefs about the right bounds for those problems. Even if better upper bounds are not possible, it is no longer a mere puzzle as in Challenge 1 that is standing in the way of higher lower bounds – substantially new techniques and frameworks must be developed.

Distance Oracles. Our first result was a mathematical advance in the understanding of the planar graph metric. Next, we use it algorithmically to achieve a new *Subset Distance Oracle* that could be an appealing choice in many applications.

A distance oracle is an encoding of a graph from which a pairwise distance can be queried efficiently. Since the seminal paper of Thorup and Zwick [73], a central subject of study in Graph Algorithms has been to understand the inherent tradeoff between the parameters of these distance oracles (see the survey by Sommer [70]): The *size* of the compression, the *query time* for returning a distance, the *error* in the answers, the *preprocessing time* to construct the compression, and so on.

Many *exact* distance oracles for planar graphs have been proposed [13, 20, 31, 35, 65, 78], mostly focusing on the tradeoff between space and query time, showing that with space s the query time can be made $\tilde{O}(n/\sqrt{s})$ [62]. Cohen-Addad, Dahlgaard, and Wulff-Nilsen [26] show that the technique of abstract Voronoi diagrams recently introduced into the field of planar graphs by Cabello [21] leads to an oracle with $O(n^{5/3})$ space and $\tilde{O}(1)$ query time, suggesting that a better tradeoff is possible. Indeed, in SODA’18 [41] the oracle was improved to $O(n^{3/2})$ space and $\tilde{O}(1)$ query time and the tradeoff was improved to s space and $\tilde{O}(n^{3/2}/s)$ query time.

To get even better tradeoffs we might allow a $(1+\varepsilon)$ error [52, 56, 71]: we can achieve very small $(1+\delta)n$ space and fast $\tilde{O}(1)$ query time. Note that $o(n)$ space is impossible in this setting (i.e., when considering distances between *all* nodes), no matter what query time we allow. However, another natural way to get better tradeoffs is to restrict our attention to a subset of the nodes. A *Subset Distance Oracle* is a small space

data structure that can efficiently return the distance between any pair of nodes from a set S of k nodes. Here, for any $k = o(n)$, e.g. $k = \sqrt{n}$, our new compression scheme suggests that a distance oracle might have $o(n)$ space.

Subset distance oracles arise naturally. In typical applications of distance oracles, one can predict that all queries will be among a subset of $k = o(n)$ nodes. Space efficiency is often a high priority. For example, if our graph is the national road network, one might be interested in a mobile app that can return the distance between any pair of bus stops.

Our second result is the first subset distance oracle with non-trivial space bounds. Notably, all previous distance oracles in the literature work equally well for weighted and unweighted graphs, while ours uses new techniques that are provably impossible for weighted graphs. Indeed, as we mentioned earlier, subset distance oracles for weighted graphs require $\Omega(\min\{k^2, n\})$ space.

THEOREM 1.2. *There is a polynomial-time algorithm that, given an unweighted undirected planar graph on n nodes and a subset S of k nodes, returns a data structure of size $O(\sqrt{k \cdot n} \cdot \log^3 n)$ words that, given any pair of nodes in S , can return their distance in $O(\min\{n^{3/4}, \sqrt{k \cdot n}\} \cdot \log^4 n)$ time.*

The main open question left by our work is whether our query time can be improved, perhaps all the way down to $\tilde{O}(1)$. This would be an essentially optimal distance oracle. But even as it is, our query time is sublinear, and our space is sublinear for any $k = o(n)$, making it an appealing choice in applications with strict space constraints.

Finally, an intriguing and wide open question is to extend (any of) our upper bounds to *directed* unweighted planar graphs. Can we accomplish Challenge 1 if we allow directed edges? Our tools heavily rely on the graph being undirected, yet it remains unclear if a higher lower bound can be proven for directed unweighted graphs. For such graphs, the current lower bound remains $\Omega(\sqrt{k \cdot n})$ and the upper bound $\tilde{O}(\min\{k^2, n\})$.

1.2 Technical Overview We exhibit the first use of the *unit-Monge* property to the algorithmic study of planar graphs. It is well known that distances in a planar graph enjoy this property, due to the non-crossing nature of shortest paths in the plane, but prior to our work, only the (*non-unit*) *Monge* property, was known to be algorithmically exploitable for planar graphs. For the past few decades, it has been heavily utilized in numerous algorithms for problems related

to shortest paths or minimum cuts in planar graphs (e.g. [18–21, 35, 40, 49, 51, 55, 60–63]), and beyond, in dozens of papers on computational geometry (e.g. [6–10, 40, 51, 53]) and pattern matching (e.g. [28, 48, 69, 75]). Meanwhile, the stronger *unit-Monge* property has only been exploited for algorithms on sequences where it has already led to several breakthroughs. For example, several variants of edit distance on strings of length n can be solved in $\tilde{O}(n)$ time [75]. These variants use the unit-Monge property to compute all pairwise distances between vertices on the boundary of an $n \times n$ grid graph. Using the weaker (non-unit) Monge property (i.e., treating the grid as an arbitrary planar graph with n^2 vertices) results in an $\tilde{O}(n^2)$ running time. We refer the reader to the 159-page monograph of Tiskin [74] for a survey on the unit-Monge property and its applications.

Recall that we want to encode the distances among k nodes in a planar graph. Let us assume that we are lucky and these k nodes are exactly the nodes of a single face of the graph, and that this face is a simple cycle. Denote the nodes appearing on the face in order $s_1, \dots, s_{k/2}, t_1, \dots, t_{k/2}$, and for simplicity assume that we only want to encode s_i -to- t_j distances. Let M be the $k/2 \times k/2$ matrix of distances so that $M[i, j] = d(s_i, t_j)$. This matrix has the Monge property: For any i, j we have that $M[i+1, j] - M[i, j] \leq M[i+1, j+1] - M[i, j+1]$. This is because the s_i -to- t_j shortest path and the s_{i+1} -to- t_{j+1} shortest path must cross. Moreover, it is *unit-Monge*, that is, $M[i+1, j] - M[i, j] \in \{-1, 0, 1\}$. This is because there is an edge between s_i and s_{i+1} and so the distances involving these nodes are always at most 1 apart.

Our gains come from the fact that unit-Monge matrices are compressible into $O(k \log k)$ bits! For non-unit Monge matrices, the construction of Gavaille et al. implies an $\Omega(k^2)$ lower bound. Another striking example for the extra power of the unit-Monge property is the fact that (a compact representation of) the distance product of two $n \times n$ such matrices can be computed in $O(n \log n)$ time [75], while for non-unit Monge matrices only $O(n^2)$ algorithms are possible. We do not use this fact in this paper, and do not know of a problem on planar graphs where this property is useful.

The main issue for us, and in general when exploiting Monge properties, is that the nodes we care about do not necessarily lie on a cycle. The simple solution is to *add* a cycle connecting our k nodes and assign weight $+\infty$ to the new edges so that they do not change the distances, or more formally, to *triangulate* the graph. After we do this, we have the Monge property, but because of the infinite weight edges, we do not have the unit-Monge property. This solution is common to all

the algorithms cited above that use the Monge property, and is quite reasonable when the graph is weighted to begin with. For unweighted graphs, on the other hand, our work proves that it is too lossy and a more involved solution leads to much better results.

At a very high-level, our approach is to use a Baker-like [15] decomposition into slices (vertices at consecutive levels of some specific BFS tree) whose boundaries are cycles, and to store distances to the slice boundaries. Observe that when we argued above that the unit Monge property holds because there is an edge between s_i and s_{i+1} , we did not require that there is also an edge between t_j and t_{j+1} . In our solution there is an edge between consecutive vertices on the boundary cycle of each slice. Therefore, even if we triangulate each slice using infinite weight edges, we can still exploit the unit Monge property when storing distances between certain vertices in a slice and the slice boundary.

The decomposition into slices is such that, after triangulation, the slices have small cycle separators. We recursively separate the vertices of the set S within each slice using small cycle separators. We store distances between separators and the slice boundary (using the unit Monge property) and between vertices of S and separators (using the fact that separators are small). Significant technical issues arise with the nesting structure of slices. This gives rise to so-called *holes* in a slice. Dealing with multiple holes requires a detailed study of additional structural properties, and a more complicated recursive solution based on these properties (see Section 3.2). In essence, we show that whenever a naïve solution does not work in the presence of multiple holes, there is one hole that can be handled efficiently using a different approach.

We believe it is very likely that other problems in unweighted planar graphs can be solved by exploiting the unit-Monge property. Our near-optimal metric compression serves as a proof of concept that this is possible. However, technical challenges might have to be overcome in each specific application. In particular, the fast distance product algorithm for unit-Monge matrices [75] appears to be a strong and relevant technique that we are so far unable to exploit for solving problems in planar graphs.

2 Preliminaries

We assume that G is connected, otherwise we handle each connected component separately. We assume basic familiarity with planar graphs and planar graph duality. We denote the primal graph by G and the dual graph by G^* . For a spanning tree T of G , we use T^* to denote the spanning tree of G^* . It is well known [77] that the set of edges of G not in T form a spanning tree T^* of

G^* . We often refer to T^* as the *cotree* of T [34]. For a spanning tree T of G and an edge e of G not in T , the *fundamental cycle* of e with respect to T in G is the simple cycle consisting of e and the unique simple path in T between the endpoints of e .

Given an assignment of nonnegative weights to the faces of G , we say that a simple cycle C is a *balanced separator* if the total weight of faces strictly enclosed by C and the total weight of faces not enclosed by C are each at most $5/6$ of the total weight.³ We often assign weights to vertices rather than to faces. Finding a balanced separator with respect to vertex weights reduces to the case of face weights (for each vertex, simply remove its weight and add it to an incident face). It is well known (see, e.g., [57]) that in triangulated planar graphs there exists a balanced separator that is a fundamental cycle assuming that no face has more than $1/2$ of the total weight (in fact, this is true for any planar graph such that T^* has maximum degree 3). For vertex-weights, if no vertex has more than $1/2$ of the total weight and the graph is triangulated and there are no self loops then by evenly transferring the weights to faces we obtain that no face receives more than $1/2$ of the total weight (because every node is incident to at least two faces) and we can invoke the face-weights version of the balanced separator. Many planar graph algorithms triangulate the graph by adding edges to ensure that short balanced cycle separators exists. The lengths of the added edges is set to be sufficiently large so as not to change distances in the graph. This is clearly not possible in unweighted planar graphs, and is one of the obstacles we will need to overcome.

2.1 The Monge and Unit-Monge properties

One of the main tools we use for succinct representation of distances in unweighted undirected planar graphs is the *unit Monge property*. We next present a sequence of lemmas that utilize this property to efficiently store distances between vertices on *cycles*. We begin with encoding distances between disjoint sets of vertices on a single face (Lemma 2.1), then encoding distances between all vertices on a single face (Lemma 2.2), and finally, encoding the distances between the vertices of two faces (Lemma 2.3).

LEMMA 2.1. *Let $C = (v_1, v_2, \dots, v_{|C|})$ be the cyclic walk of a face of a planar graph partitioned into two parts $C_1 = (v_1, v_2, \dots, v_\ell)$ and $C_2 = (v_{\ell+1}, v_{\ell+2}, \dots, v_{|C|})$. Then, for any subset C'_2 of C_2 ,*

³It is more usual to require that the total weight is at most either $2/3$ or $3/4$. However, in our particular application $5/6$ turns out to be more convenient.

all distances between vertices of C_1 and vertices of C'_2 can be encoded in $\tilde{O}((|C_1| + |C'_2|) \log |C|)$ bits.

Proof. Let $C'_2 = \{v_{p_1}, v_{p_2}, \dots, v_{p_s}\}$. We define an $\ell \times s$ matrix M such that $M[i, j]$ equals the distance in G between v_i and v_{p_j} . The matrix M is Monge, that is $M[i + 1, j] - M[i, j] \leq M[i + 1, j + 1] - M[i, j + 1]$ for any $i \in [1, \ell - 1]$ and $j \in [1, s - 1]$. This is because the shortest v_i -to- v_{p_j} and v_{i+1} -to- $v_{p_{j+1}}$ paths must necessarily cross. Furthermore, the matrix M is unit-Monge, that is $M[i + 1, j] - M[i, j] \in \{-1, 0, 1\}$ for any $i \in [1, \ell - 1]$ and $j \in [1, s]$, because there is an edge (v_i, v_{i+1}) . Consequently, for any $i \in [1, \ell - 1]$, the sequence of differences $M[i + 1, j] - M[i, j]$ is nondecreasing and contains only values from $\{-1, 0, 1\}$, so can be encoded by storing the positions of the first 0 and the first 1. Storing these positions for every $i \in [1, \ell - 1]$ takes $O(\ell \log s)$ bits. To encode M , we additionally store $M[0, j]$ for every $j \in [1, s]$ using $O(s \log |C|)$ bits. \square

We stress that in the statement of Lemma 2.1, the vertices C_1 must be consecutive on the face, but the vertices of C'_2 need not. This suffices for the unit Monge property to apply. We will critically use this fact in our construction.

LEMMA 2.2. *Let $C = (v_1, v_2, \dots, v_{|C|})$ be the cyclic walk of a face of a planar graph. Then, all distances between vertices of C can be encoded in $O(|C| \log^2 |C|)$ bits.*

Proof. We recursively encode all distances between vertices from a contiguous fragment of C using Lemma 2.1. We start with the whole $v_1, v_2, \dots, v_{|C|}$. To encode the distances between all vertices v_i, v_{i+1}, \dots, v_j , where $i < j$, we set $m = \lfloor (i + j)/2 \rfloor$ and proceed as follows:

1. Recursively encode the distances between all vertices v_i, v_{i+1}, \dots, v_m .
2. Recursively encode the distances between all vertices $v_{m+1}, v_{m+2}, \dots, v_{|C|}$.
3. Apply Lemma 2.1 with $C_1 = (v_i, v_{i+1}, \dots, v_m)$ and $C'_2 = \{v_{m+1}, v_{m+2}, \dots, v_j\}$.

The total size of the encoding is described by the recurrence $T(s) = O(s \log |C|) + 2T(s/2)$, hence solves to $O(|C| \log^2 |C|)$. \square

LEMMA 2.3. *Let $C_{ext} = (v_1, v_2, \dots, v_{|C_{ext}|})$ and $C_{int} = (u_1, u_2, \dots, v_{|C_{int}|})$ be the cyclic walks of two faces of a planar graph. Then, all distances between a prefix C'_{ext} of C_{ext} and any subset C'_{int} of C_{int} can be encoded in $\tilde{O}((|C'_{ext}| + |C'_{int}|) \log n)$ bits.*

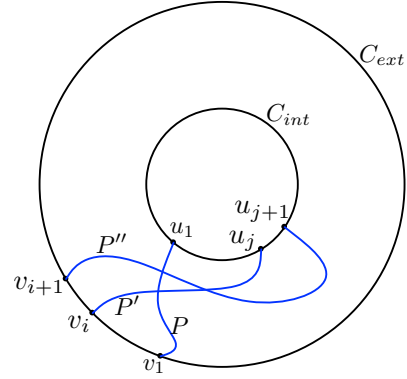


Figure 1: The Monge property in Lemma 2.3.

Proof. We first choose a shortest path P between C'_{ext} and C_{int} and let v_i and u_j be its endpoints. We make an incision along P and apply Lemma 2.1 to encode the distances between C'_{ext} and C'_{int} corresponding to shortest paths that do not cross P using $O((|C'_{ext}| + |C'_{int}|) \log n)$ bits of space (the incision makes the size of the face we apply Lemma 2.1 to $O(n)$). It remains to encode distances corresponding to shortest paths that do cross P . Without loss of generality P connects v_1 and u_1 . We orient C_{ext} and C_{int} so that after making an incision along P the vertices v_2 and u_2 are adjacent to the endpoints of P .

Consider a shortest path P' from v_i to u_j crossing P , see Figure 1. Because both P and P' are shortest paths, P' can be assumed to cross P exactly once. Let P'' be a shortest path v_{i+1} to u_{j+1} crossing P . Similarly, P'' can be assumed to cross P exactly once. We claim that P' must cross P'' . Otherwise, by considering an incision along P' we can conclude that P'' crosses P an even number of times but this is a contradiction. Therefore, any such P' and P'' must cross. This means that the matrix M , where $M[i, j]$ is set to be the length of a shortest path between v_i and u_j crossing P once, is Monge. That is,

$$M[i + 1, j + 1] + M[i, j] \geq M[i + 1, j] + M[i, j + 1].$$

Additionally, $M[i, j] - M[i + 1, j] \in \{-1, 0, 1\}$ because (v_i, v_{i+1}) is an edge. We can hence apply the reasoning from Lemma 2.1 to encode M using $O((|C'_{ext}| + |C'_{int}|) \log n)$ bits. \square

3 The Encoding

Our encoding is based on decomposing the input graph G into slices. To define the slices, recall the face-vertex incidence graph $FV(G)$ of a planar graph G : It has a vertex for every vertex v of G and a vertex for every face f of G , and if a vertex v of G is incident to a face f

of G then there is an edge between their corresponding vertices in $FV(G)$.

We run a breadth-first search in $FV(G)$, starting from the node representing the infinite face of G . After every even number of steps, the yet unexplored part of the graph can be decomposed into a number of connected components, the boundary of each being a simple cycle. More formally, we assume that the infinite face of G is a triangle by enclosing the whole graph in a triangle, which is connected to one of the original vertices with a single edge. We define the level of a face f or a vertex v of G to be its depth in the BFS tree of $FV(G)$. Thus, e.g., the level of the infinite face of G is zero, and the level of the vertices incident to the infinite face of G is 1. For each even integer $i \geq 2$, consider the set of connected components of the subgraph of G^* induced by the faces (of G) with level at least i . We define $\mathcal{K}_{\geq i}$ to be the set of subgraphs of G induced by the faces (of G) in each of these connected components of G^* . We call each element $K \in \mathcal{K}_{\geq i}$ a level- i component. We use a tree \mathcal{K} called the *component tree* of G to capture the nesting of level components. The nodes of \mathcal{K} are the level components of G . A level component K is an ancestor of a level component K' in \mathcal{K} if the set of faces in K contains the set of faces in K' . Since we assume that the infinite face of G is a simple triangle, \mathcal{K} is indeed a tree whose root is the component corresponding to the set of all faces of G except for the infinite face.

The boundary of a component K is the set of edges that are incident to a face in K and to a face not in K . It is not difficult to see that the boundary of each component K is a simple cycle in G , and that the boundaries of different components are edge-disjoint. See [54,57] for these and other properties of components and the component tree. For a node $k \in \mathcal{K}$, we associate k with the boundary cycle C_k of the level component represented by k , and define the cost of k denoted $\text{cost}(k)$ to be the length of C_k . For example, for the root r of \mathcal{K} we have that C_r is a triangle and that $\text{cost}(r) = 3$.

LEMMA 3.1. *For any $w \geq 1$, there exists $\delta \in [0, w)$ such that the total cost of all nodes of \mathcal{K} at depth $\delta, \delta + w, \delta + 2w, \dots$ is $O(n/w)$.*

Proof. $\sum_{v \in \mathcal{K}} \text{cost}(v) = O(n)$ because cycles corresponding to the nodes of \mathcal{K} are pairwise edge disjoint. Let $S(\delta)$ consist of all nodes of \mathcal{K} at depth $\delta, \delta + w, \delta + 2w, \dots$. Then $S(\delta) \cap S(\delta') = \emptyset$ for $\delta \neq \delta'$ and $\sum_{\delta \in [0, w)} \sum_{v \in S(\delta)} \text{cost}(v) = O(n)$, so there exists $\delta \in [0, w)$ such that $\sum_{v \in S(\delta)} \text{cost}(v) = O(n/w)$ as claimed. \square

To define the slices we apply Lemma 3.1 and call the nodes of \mathcal{K} at depth $\delta, \delta + w, \delta + 2w, \dots$ *marked*. The root

of \mathcal{K} is also marked. Then, for every marked node $v \in \mathcal{K}$, the slice of v is the subgraph of G enclosed by C_v and not strictly enclosed⁴ by C_u for any marked descendant u of v . The embedding of slices is inherited from the embedding of G . Thus, the boundary of the infinite face of the slice s of v is C_v . The cycle C_v is also called *the boundary of the slice s* . Each cycle C_u corresponding to a marked descendant u of v such that there are no other marked nodes on the v -to- u path becomes a face in the slice s . Such a face is called a *hole* of s , and C_u is called *the boundary of the hole*. We sometimes refer to the boundary of a slice as its external boundary, to emphasize that we do not mean the boundary of the holes of the slice. Note that, by definition, C_u is the external boundary of the level component that is embedded in the hole u . Because the total cost of all marked nodes is $O(n/w)$ and the cost of the root is 3, the total size of all boundaries in all slices is $O(n/w)$. Additionally, by construction, for any slice s , a breadth-first search of $FV(s)$, the face-vertex incidence graph of s , starting at the infinite face of s , terminates after $O(w)$ iterations and every hole is a leaf in the obtained breadth-first search tree.

By definition of slices, each slice contains faces and vertices at $O(w)$ consecutive levels. We would like to use in our solution short (i.e., $O(w)$) fundamental cycle separators within each slice. However, the diameter of a slice is not necessarily $O(w)$ because face sizes may be large. To deal with this issue we triangulate the faces so that a BFS tree of a slice will have depth $O(w)$, and will be consistent with the BFS tree of $FV(s)$.

Let T_{FV} be the BFS tree of $FV(s)$. We add edges to s to make it a triangulated graph s' in the following manner. For any two vertices v and u that are incident to the same face in s , and v is a grandparent of u in T_{FV} , and vu is not an edge in s , we add vu as an artificial triangulation edge. Adding these edges can be done consistently with the embedding of s because the path in T_{FV} can be embedded on the same plane as s such that s and T_{FV} only intersect at vertices of s . See Figure 2. We introduce an artificial vertex v_s embedded in the infinite face of s and triangulate the infinite face of s by adding edges between v_s and every vertex of the infinite face of s . Similarly, we triangulate each hole h of s by introducing an artificial vertex v_h , embedded in h , and adding edges between v_h and every vertex on the boundary of h . Any remaining non-triangulated faces are triangulated arbitrarily.

⁴A vertex or an edge x is *enclosed* by a cycle C if x is incident to a face enclosed by C . If x is enclosed by C but $x \notin C$ then x is said to be *strictly enclosed* by C .

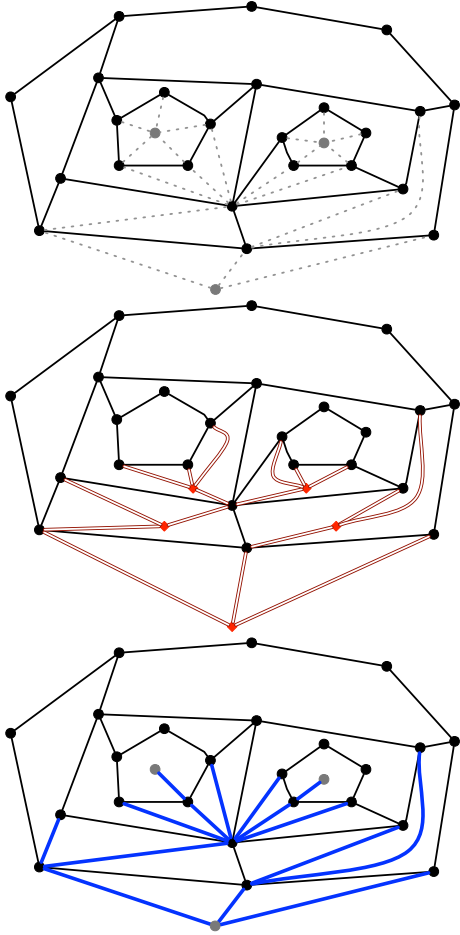


Figure 2: Triangulating a slice. The vertices and edges of a slice s with two holes are shown in solid black. In this example $w = 2$. Edges of T_{FV} , the BFS tree of $FV(s)$ are shown in double red lines. Only some of the edges of T_{FV} are shown to avoid clutter. Artificial triangulation edges are shown in dashed gray. The BFS tree T_s of the triangulation of s is shown in blue.

LEMMA 3.2. *a BFS of s' starting from v_s produces a spanning tree T_s with the property that vertex v is the parent of vertex u in T_s if and only if v is the grandparent of u in T_{FV} .*

Proof. By construction of s' , for every grandparent to grandchild path in T_{FV} there is a corresponding edge in s' . Hence, there exists a BFS tree T_s rooted at the artificial vertex v_s that satisfies the statement of the lemma. Note that all the artificial vertices embedded in holes of s are leaves of T_s , and hence satisfy the statement of the lemma vacuously. \square

Let T_s be the BFS tree of s' from Lemma 3.2. The utility of using artificial triangulation edges in s'

is that now any fundamental cycle C w.r.t. T_s consists of two paths in T_s , each consisting of $O(w)$ vertices. However, C may use edges that are not original edges of s (i.e., artificial triangulation edges). We do not want to consider such edges when dealing with distances, because distances in s' differ from distances in s . To this end we use the notion of a Jordan curve. A Jordan curve in s is an embedded curve that intersects the embedding of s only at vertices of s . Since the embedding of the triangulation s' is consistent with that of s , each path in T_s is a Jordan curve in s . We say that T_s is a *Jordan tree* in s . In particular, any fundamental cycle w.r.t. T_s is a Jordan cycle (closed Jordan curve) in s . We next describe how the tree T_s can be used to recursively decompose s into subgraphs called *regions*.

A region R is a subgraph of s (not of s'). The boundary of R is defined as the set of vertices of R that either belong to the external boundary of s , or to the boundary of a hole of s , or are incident (in s) to both an edge in R and to an edge not in R . Thus, for example, the boundary of the region consisting of the entire slice s consists of the external boundary C_{ext} of s and of the boundaries of all the holes of s . Let R be a region. Let C be a fundamental cycle w.r.t. T_s . The tree T_s may contain edges that are not edges of R (either because they are triangulation edges, or because they are edges of s that do not belong to the region R). Since the embedding of T_s is consistent with the embedding of any subgraph of s , C is a Jordan cycle in R . The operation of separating R using C yields two subgraphs. One is the subgraph induced by the faces of R strictly enclosed by C and the other is the subgraph induced by the faces of R not strictly enclosed by C . This view of T_s as a Jordan tree in any region allows us to reuse the same tree T_s throughout the recursive decomposition. Note that, if R' is one of the regions obtained by separating R using C , then the boundary of R' is larger than the boundary of R by $O(w)$ vertices. This is true since the Jordan cycle C consists of $O(w)$ vertices of R . Also note that the boundary vertices of R' that belong to C lie on a single face of R' . However, they are not necessarily consecutive on that face (they were consecutive in the Jordan tree T_s , but this tree contains triangulation edges, whereas R' does not, so in R' such consecutive vertices lie on the same face, but are not necessarily adjacent). We will nonetheless be able to use the unit Monge property because, as pointed out earlier, Lemmas 2.1 and 2.3 only require one of the subsets to be consecutive along a face.

This recursive process can be described by a binary tree \mathcal{T}_s . Each node v of \mathcal{T}_s corresponds to a region (subgraph) R_v of s . The root of \mathcal{T}_s is the entire slice s . Each non-leaf node v of \mathcal{T}_s is associated with a (Jordan)

fundamental cycle separator of T_s , denoted Sep_v , whose precise choice is described in the next subsection. The regions of the two children of v are the regions obtained by separating R_v using the Jordan cycle Sep_v .

3.1 The simplified case of a single hole We begin with the simplified case, in which we assume that each slice has a single hole. This is the case, for example, when the input planar graph is a grid (with possibly subdivided edges).

First we use Lemmas 2.2 and 2.3 to store, for each slice s with external boundary C_{ext} , and a single hole h with boundary C_h the following distances. The **boundary-to-boundary** distances: the distances (in s) among the vertices of C_{ext} , and the **hole-to-boundary** distances: the distances (in s) between the vertices of C_{ext} and the vertices of C_h .

Boundary-to-boundary and hole-to-boundary distances encode distances “between slices”. We also need to encode distances “within slices”. We will use the fact that s has a spanning tree of depth $O(w)$ to decompose s into regions, each containing a single distinguished node (i.e., a single node from S), using separators that consists of $O(w)$ vertices. Then we can afford to store, for each distinguished node, its distance to the entire boundary of its region, and, using the unit-Monge property, to also store the distances between the $O(w)$ vertices on the separator used to obtain each region R to the vertices of C_{ext} and C_h (i.e., the boundary of s) that belong to R . These distances will suffice for reconstructing the distance between any pair of distinguished nodes.

Let S_s denote the set of distinguished vertices in slice s . We use fundamental (Jordan) cycle separators w.r.t. the tree T_s to recursively divide s into regions, until each region contains a single distinguished vertex. Consider a step in the recursion involving a region R_v that corresponds to a node v in the recursive decomposition tree T_s . We separate R_v into two subregions by taking Sep_v to be a fundamental cycle separator w.r.t. T_s that balances the number of distinguished vertices in R_v (i.e., assigning unit weight to each distinguished vertex in R_v and zero weight to all other vertices). Note that, since we use balanced separators, the depth of the recursion tree T_s is $O(\log |S_s|) = O(\log n)$. Recall that the fundamental cycle separators w.r.t. T_s do not cross each other, and, by construction of T_s in Lemma 3.2, each fundamental cycle separator crosses each of the external boundary of s and the hole of s at most twice. Therefore, the boundary of each region R_v contains $O(\log n)$ vertex disjoint maximal subpaths of C_{ext} , and $O(\log n)$ vertex disjoint maximal subpaths of C_h . Furthermore, for any two regions at the same depth in T_s , the maxi-

mal subpaths of these two regions may share only their endpoints.

At the step of the recursive decomposition corresponding to node $v \in T_s$ with separator Sep_v and two children u, w , we store **S-to-separator** distances: explicitly store the distances (in R_v) between every vertex of S in R_v and every vertex of Sep_v , **separator-to-boundary** distances and **separator-to-hole** distances: for $i \in \{u, w\}$, the distances (in R_i) between every vertex of Sep_v and every maximal subpath of C_{ext} or C_h on the boundary of R_i , using Lemma 2.1 or Lemma 2.3 (depending whether they lie on the same face or on two distinct faces of R_i). Finally, for every leaf $v \in T_s$, we store **S-to-boundary** distances and **S-to-hole** distances: the distance between the unique distinguished vertex in R_v to every vertex of C_{ext} or C_h on the boundary of R_v .

Analysis. We first show that the total space is $\tilde{O}(\sqrt{k} \cdot n)$, and then show that the distances between any pair of vertices in S can be recovered using just the information we stored. Since the total size of all slice boundaries is $O(n/w)$, storing the boundary-to-boundary distances and the hole-to-boundary distances takes $\tilde{O}(n/w)$ using Lemmas 2.2 and 2.3. Since the depth of T_s is $O(\log n)$, each vertex of S_s belongs to $O(\log n)$ regions in the decomposition of s . Since, in addition, $|Sep_v| = O(w)$ for every $v \in T_s$, the total space required for storing the S -to-separator distances is $\tilde{O}(k \cdot w)$. Consider a region R of a slice s . Recall that the vertices of C_{ext} (C_h) that belong to R lie on $O(\log n)$ vertex disjoint maximal subpaths of C_{ext} (C_h). The endpoints of each such maximal subpath may belong to another region R' at the same depth in T_s . Therefore, R shares $O(\log n)$ vertices of C_{ext} (C_h) with other regions at the same depth in T_s . Finally, recall that the number of regions of s is $\tilde{O}(|S_s|)$. Therefore, using Lemma 2.1 or Lemma 2.3, the total space for storing the separator-to-boundary and separator-to-hole distances is $\tilde{O}(n/w + k + k \cdot w)$. In more detail, let c_i be the total number of slice/hole boundary vertices in the i -th slice. Then, in every slice every boundary/hole vertex that is not an endpoint of a maximal subpath contributes at most once at each level of recursion. At each level, we have at most k recursive calls, so at most $O(k \log n)$ maximal subpaths and at most k fundamental cycle separators. Therefore, the total space is $O((\sum_i c_i + k \log n + k \cdot w) \log n) = \tilde{O}(n/w + k + k \cdot w)$. Storing S -to-boundary distances and S -to-hole distances at the leaves of the recursion tree requires total $\tilde{O}(k+n/w)$ bits since each boundary or hole vertex belongs to exactly one leaf region, except for $O(k \log n)$ vertices (endpoints of maximal subpaths). Choosing $w = \sqrt{n/k}$ proves the space bound.

Finally, we prove that the distances between any pair of vertices in S can be recovered using just the information we stored. For any $x, y \in S$, if a shortest x -to- y path does not leave s then $x, y \in S_s$, and the distance can be obtained using the S -to-separator distances stored in the lowest common ancestor of the regions of x and y in \mathcal{T}_s . Otherwise, let P be a shortest path between vertices $x \in S_{s'}$ and $y \in S_s$ (where s' is either s or, wlog, enclosed by the hole of s). Let $P[i, j]$ denote the subpath of P between vertices i and j . Let v be the first vertex of P that belongs to the boundary of s' or to the boundary of a hole of s' . If $P[x, v]$ contains some vertex of a fundamental cycle separator used in processing s' , let u be the last vertex of P that belongs to the earliest such separator. If u does not exist, then the length of $P[x, v]$ is stored as an S -to-boundary or an S -to-hole distance. If u exists then the length of $P[x, u]$ is stored as a S -to-separator distance, and the length of $P[u, v]$ is stored as a separator-to-boundary or separator-to-hole distance. Let w be the last vertex of P that belongs to the boundary of s . The length of $P[v, w]$ can be computed from boundary-to-boundary and hole-to-boundary distances since $P[v, w]$ can be decomposed into subpaths between boundary vertices of slices. The length of the suffix $P[w, y]$ can be computed in a similar manner to that of the prefix $P[x, v]$.

3.2 The general case A difficulty that arises in the presence of multiple holes is that since the number of holes is not bounded by a constant, we cannot afford to store distances involving holes. For example, storing hole-to-boundary distances between the external boundary C_{ext} of a slice s and the boundary of each hole of s requires $\Omega(|C_{ext}|) = \Omega(n/w)$ bits per hole. Since the number of holes can be $\Omega(n)$, the total space could be $\Omega(n^2/w)$.

The role of storing distances involving boundaries of holes was to allow the recovery of distances to distinguished vertices enclosed in these holes. We modify our approach for processing a slice s to take into account the distinguished vertices enclosed in holes of s as well as the distinguished vertices in s itself. As in the single hole case, the slice s will be recursively divided using fundamental cycle separators. For any region R encountered along the recursive process, let S^R denote the subset of the distinguished vertices in R , as well as those enclosed by any hole in R . Thus, for example, S^s is the set of all vertices in S that are enclosed (in G) by the external boundary of slice s . We say that a Jordan cycle separator C of a region R is *good* if it is balanced w.r.t. S^R and does not go through any hole of R . The problem with Jordan separators that go through some hole h is that they partition the distinguished

vertices enclosed by h in an unspecified way since these distinguished vertices are not represented in R . It is not hard to see that if a good separator always exists then we do not need to store any distances involving holes.

In reality we cannot always find a good separator. Consider, for example, the case where some hole h of a region R encloses most of the vertices of S^R . Clearly, a separator that is balanced w.r.t. S^R must go through h . Thus, there is no good separator in such a case. We show, however, that we can always either find a good separator, or there exists some hole (which we call a *disposable* hole) that can be dealt with in a special way. This is reminiscent of recursive procedures based on heavy path decomposition, where heavy nodes (disposable holes in our case) are treated differently than light ones. We guarantee that, in either case, each resulting subregion contains only a constant fraction of S^R , so the depth of the recursion is $O(\log n)$. We next explain the details.

Good separators and disposable holes. Let R be a region. We define the weight of each vertex v of R to be 1 if v is a distinguished vertex. For each hole h of R , we define the weight of the artificial vertex v_h embedded in h to be the number of distinguished vertices strictly enclosed (in the whole graph G) by the boundary of h . All other vertices are assigned weight zero.

Recall that a cycle separator is *good* if it does not go through any hole. We would like to separate R using a good fundamental cycle separator C_e of some edge e w.r.t. T_s . If we can find such separator C_e where e is not incident to v_h for some hole h , then C_e is a good separator (since the vertices v_h are leaves of the spanning tree T_s). Otherwise, we must separate R with a fundamental cycle separator that goes through holes. We next define *disposable* holes, and then show that we can allow the fundamental cycles to go through such holes.

Let k be a node (level component) in \mathcal{K} . Let C_k be the boundary cycle of k . Let e be an edge of C_k . Note that $e \notin T_s$. This is because both endpoints e have the same level, so, by Lemma 3.2, neither can be the parent of the other in T_s . Let f, g be the endpoints of e in the dual graph, such that f is a face in k and g a face not in k . Since $e \notin T_s$, e is in the cotree T_s^* . Consider breaking T_s^* into two subtrees by deleting e . We say that the edge e is *light* if the subtree of T_s^* that contains g has weight at most $W/2$ where W is the total weight of the vertices of R . Note that we defined weights of primal vertices, whereas the vertices of T_s^* are primal faces. To define face weights, evenly redistribute the weight of each vertex among all of its incident faces. There is an equivalent, primal view of light edges: The Jordan cycle

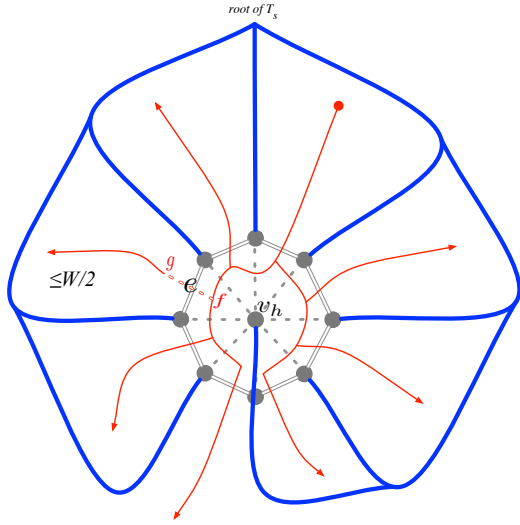


Figure 3: An illustration of a region with a disposable hole. The edges of a boundary of a hole h are shown in double-line grey. Since the boundary of a hole is a level cycle, none of the edges of the boundary of h belongs to the spanning tree T_s (blue). The artificial triangulation vertex v_h of h and the triangulation edges (grey dashed) are shown. The cotree T_s^* is shown in thin red. Suppose that the number of distinguished nodes enclosed by h is at least $W/2$ (so the weight of v_h is at least $W/2$). Then, for any edge e of the boundary of h , the part of $T_s^* \setminus e$ that does not contain faces of h weighs at most $W/2$, so e is a light edge and h is a disposable hole.

C_e partitions R into two subgraphs, exactly one of which contains the faces of the level component corresponding to k . We say e is *light* if the weight of the subgraph that does not contain the level component k is at most half the weight of R . We say that a level component k is *disposable* in region R if there are boundary edges of k in R , and if every edge e of the boundary of k that is also in R is light. Note that, in particular, this definition applies to holes (since holes are level components). See Figure 3.

Before showing why disposable holes exist and that they are useful, we first mention a simple property of T_s^* and use it to prove the existence of disposable holes.

PROPERTY 3.1. *The cotree T_s^* enters each level component exactly once.*

Proof. The spanning tree T_s is monotone with respect to node levels. Thus, if e is an edge of the boundary of a level component k , then one of the components of $T_s^* \setminus e$ contains no other faces, vertices or edges of k . See Figure 4 for an illustration. \square

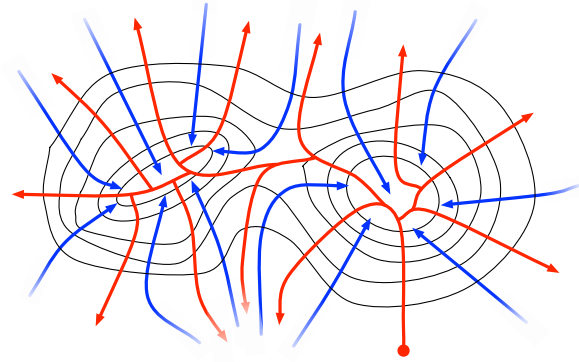


Figure 4: An example of the interaction between the spanning tree T_s (blue), the cotree T_s^* (red), and boundary of level components (black cycles). Since T_s is monotone with respect to levels, T_s^* enters each level component exactly once.

LEMMA 3.3. *If a region R contains more than one vertex with non-zero weight, then there exists either a good balanced fundamental cycle separator or a disposable hole in R .*

Proof. Let W be the total weight of vertices in R . Consider the component tree \mathcal{K} . Let u be a deepest disposable component in \mathcal{K} such that C_u has an edge in R . If u is a hole of R then we found a disposable hole, and we are done. Otherwise, we next show that there exists a good separator.

Let u_1, u_2, \dots, u_d be the children of u in \mathcal{K} (if there is no disposable component in R , then define u to be R , C_u to be the external boundary of R , and let u_1, \dots, u_d be the set of rootmost components in \mathcal{K} such that C_{u_i} has an edge in R). Since none of the u_i 's is disposable, for each u_i there exists exactly one boundary edge $e_i = (f_i, g_i)$ (here we wrote e_i as a dual edge, and f_i is the endpoint of e_i that belongs to u_i), such that the subtree of $T_s^* \setminus e_i$ that contains g_i has weight at least $W/2$. Consider the following two phase process (see Figure 5 for an illustration): Let $T_0^* = T_s^*$. If T_i^* contains more than a single face of some u_j (in which case it must contain all faces of u_j by Property 3.1), then T_{i+1}^* is obtained from T_i^* by rooting T_i^* at g_j and deleting all the strict descendants of f_j in T_i^* , so that f_j becomes a leaf. The weight assigned to f_j in T_{i+1}^* is the total weight of all the vertices in the deleted subtree. Thus, the weight of T_{i+1}^* remains W , and, by definition of e_j , the weight of f_j is at most $W/2$. The first phase terminates when T_i^* contains at most one face (f_j) from each u_j . In the second phase, while T_i^* contains an edge e of C_u that is not a leaf edge of T_i^* , then T_{i+1}^* is

obtained from T_i^* by rooting T_i^* at the endpoint g of e that belongs to u , and deleting all the strict descendants of the other endpoint f of e in T_i^* , so that f becomes a leaf. Similarly to the first phase, the weight of f in T_{i+1}^* is set to the total weight of all the vertices in the deleted subtree. Since u is disposable, the weight of f is at most $W/2$.

Let T_t^* be the resulting tree. Since T_t^* contains at most one face from each u_i , T_t^* contains no triangulation edges of a hole (both endpoints of a triangulation edge of a hole belong to the hole). Furthermore, the total weight of T_t^* is W , and every leaf of T_t^* created during the two phase process has weight at most $W/2$ (by definition). For the remaining nodes of T_t^* , the degree is at most 3 and the weight is also at most $W/2$, because the original weights in T_s^* are at most $W/2$ (otherwise, the node corresponds to a hole of weight at least $W/2$ that is, by definition, disposable, and we are done). Therefore, there exists an edge e whose deletion from T_t^* results in two trees, none of which weighs more than $5W/6$. By construction of the weights of T_t^* , the balance of the fundamental cycle of e w.r.t. T_s is exactly the ratio of the weights of the subtrees obtained by deleting e from T_t^* . Therefore, the fundamental cycle C_e of e w.r.t. T_s is a balanced Jordan cycle separator. Since no edge of T_t^* is a triangulation edge of a hole, we have that C_e is a good separator. \square

With this structural lemma we can now describe our oracle. Consider a slice s and let G_s be the subgraph of G enclosed by the boundary of s . The goal of processing slice s is to store information (distances) so that the following distances (in G_s) can be recovered from the information stored for all slices contained in G_s .

1. The distance between any two distinguished nodes in G_s ,
2. The distance between any distinguished node in G_s and any vertex on the boundary of s .
3. The distance between any two vertices on the boundary of s .

Encoding this information for all slices guarantees that distances between the distinguished vertices in the whole graph are captured.

The encoding. To process a slice s , we first encode **boundary-to-boundary** distances: the distances (in G_s) between vertices on the boundary of s using Lemma 2.2. We then triangulate s and define its spanning tree T_s using Lemma 3.2.

Next, we recursively separate s using fundamental cycle separators. The initial region R is the entire slice s . Its boundary is the external boundary C_{ext} of

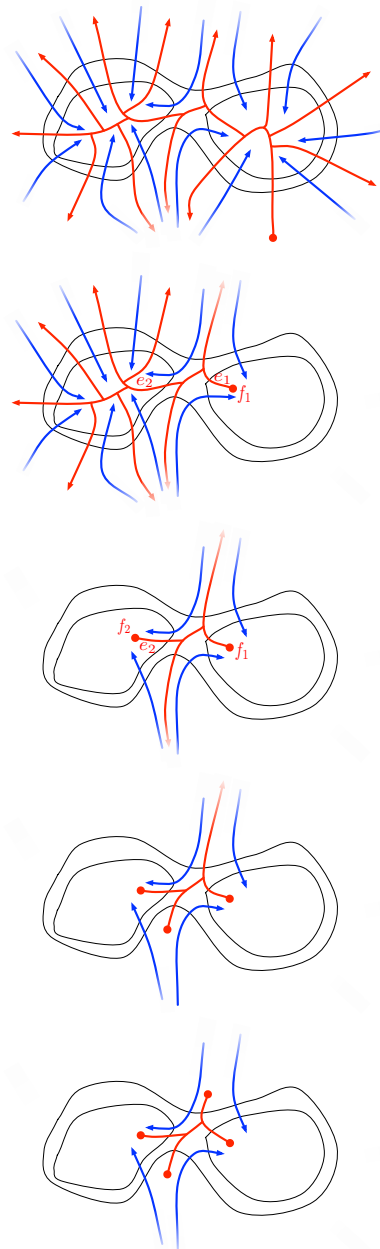


Figure 5: Illustration of the process of constructing the cotree T_t^* according to which a good separator is found. Three level boundaries are shown (black cycles). The external one is C_u , the two cycles enclosed by C_u are C_{u_1} and C_{u_2} . The subfigures show the process of splitting the cotree T_s^* , first at e_1 , then at e_2 . In the second phase the cotree is further split at the two remaining edges of C_u . The resulting tree T_t^* is an induced subtree of T_s^* in which a balanced edge-separator can be found. Since none of the edges of T_t^* has both endpoints in any u_i , none of the edges of T_t^* are triangulation edges of a hole.

s. A region R is separated into subregions obtained by cutting R along some fundamental cycle separator C w.r.t. T_s . Since we only use fundamental cycle separators w.r.t. the same tree T_s , the separators never cross. Hence, the boundary of each new region R' consists of the contiguous portion of C that belongs to R , and possibly portions of the boundary of R . Since C crosses C_{ext} at most twice (at most once for each of the two paths in the fundamental cycle C), the number of contiguous maximal fragments of C_{ext} in the boundary of R' is at most one plus the number of such fragments in the boundary of R . Consequently, the number of contiguous maximal fragments of C_{ext} in the boundary of any region is bounded by the depth of the recursion, which we will show is $\tilde{O}(1)$.

We now explain how to choose the fundamental cycle separator C with which we separate R . This is achieved using two interleaving recursive processes. We refer to the first one as the outer recursion, and to the second one as the hole elimination recursion. In a step of the outer recursion we apply Lemma 3.3.

If we find a good balanced fundamental cycle separator C , then we use it to separate the region R . Every vertex in S^R explicitly stores **S-to-separator** distances: its distance (in G_s) to every vertex of C . In addition, for each subregion R' , for each contiguous maximal fragment b_i of C_{ext} in R' , we encode **separator-to-boundary** distances: the distances (in R') between b_i and C using Lemma 2.1 or Lemma 2.3 (depending on whether the vertices of the separator C and the vertices of b_i lie on a single or two faces of R'). Then, we call the outer recursion recursively for each subregion R' . The outer recursion terminates when there is at most one vertex with positive weight in the current region R . If the only remaining object is an artificial vertex v_h , we apply Lemma 2.3 to encode **hole-to-boundary** distances: the distances (in R) between the boundary C_h of h and b_i , for each contiguous maximal fragment b_i of C_{ext} in R . If the only remaining object is a distinguished vertex u , we store **S-to-boundary** distances: the distances (in G_s) from u to every vertex of every b_i . If the current region R contains no vertices with positive weight, the outer recursion terminates.

If, on the other hand, we found a disposable hole h , we store **hole-to-boundary** distances: distances between the boundary C_h of h and every contiguous maximal fragment b_i of C_{ext} in R . The weight of the artificial vertex v_h is set to zero. This reflects the fact that for the rest of the processing of s , distinguished vertices enclosed by the hole h will not be treated individually and directly, but rather by encoding distances involving the vertices of C_h . From this point on, vertices of S inside h are no longer considered vertices of

S^R . We then call the hole elimination process for the hole h in region R (see Figure 6). In a single step of the hole elimination recursion, a region R is separated using a fundamental cycle separator C w.r.t. T_s that is balanced w.r.t. the number of vertices of C_h in R (i.e., a weight 1 is assigned to each vertex of C_h and 0 to all other vertices). Note that C is necessarily a fundamental cycle w.r.t. T_s of some triangulation edge that is incident to v_h . The boundary of each of the two resulting regions contains a single contiguous portion of C_h consisting of roughly half the vertices of C_h in R . Similarly to the single hole case, we store **S-to-separator** distances: distances (in G_s) from every vertex of S^R to every vertex of C . For each subregion R' obtained by separating R along C , for each contiguous fragment b_i of C_{ext} in R' , we encode **separator-to-boundary** distances: the distances (in R') between b_i and C using Lemma 2.1 or Lemma 2.3, and **separator-to-hole** distances: the distances (in R') between C and the single contiguous fragment of C_h that belongs to R' using Lemma 2.1. We then apply the hole elimination process recursively to each subregion R' . It terminates when the current region R contains at most two consecutive vertices of C_h , or when it contains at most one distinguished vertex. When this happens, we continue with the outer recursion on R .

We next prove that the total depth of the entire recursive procedure is $O(\log^2 n)$.

Analyzing the recursion depth. We begin with the initial region R_0 being the entire slice. In a single step of the outer recursion, if we find a good separator then we use it to separate the current region R_0 thus decreasing the weight of each resulting region R by a constant factor. If however we do not find a good separator, then we apply the hole elimination process on a disposable hole h of the current region R_0 . Since $|C_h| = O(n)$, and since every recursive call to the hole elimination process decreases the number of nodes of C_h by half, we get that after $O(\log n)$ recursive calls the hole elimination process terminates, with each resulting region R containing only two nodes of C_h . Observe that these two nodes must be adjacent on C_h (see Figure 6). Let e be the edge between them and let C_e be the fundamental cycle of e w.r.t. T_s . Since h is disposable, the weight of the region R' obtained by separating R_0 using C_e is at most half the weight of region R_0 . Since $R = R' \cup \{v_h\}$ and since the weight of v_h is zero this means that the weight of R is at most half the weight of R_0 . We conclude that every $O(\log n)$ consecutive recursive calls the total weight of a region decreases by a constant factor. This shows that the depth of the recursion is $O(\log^2 n)$.

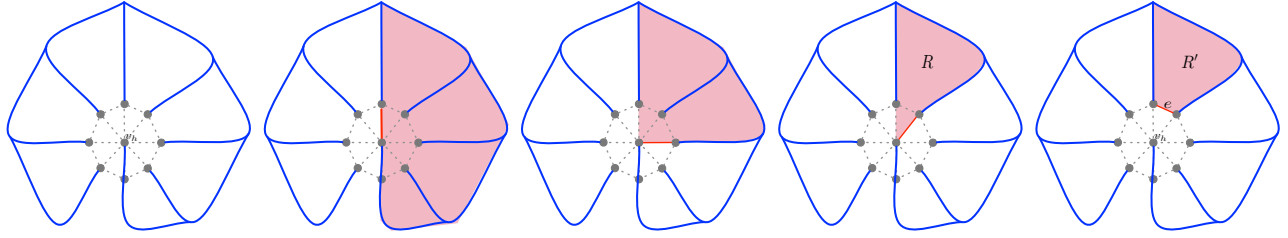


Figure 6: Illustration of the process of eliminating a disposable hole. A disposable hole h with artificial vertex v_h in a region R_0 is shown in top-left. The spanning tree T_s is indicated in blue. R_0 is recursively separated using fundamental cycle separators that are balanced w.r.t. the number of nodes of the boundary C_h of h . The elimination process finishes (bottom-left) when the current region R contains only two consecutive vertices of C_h , which are the endpoints of some edge e . This region R differs from the the region R' obtained by separating R_0 using the fundamental cycle of e w.r.t. T_s by a single vertex (v_h).

Correctness. We next prove that the distance between any two distinguished vertices in S can be recovered from our encoding.

LEMMA 3.4. *The length of a shortest path P in G_s from any $x \in S_s$ to any $y \in C_{ext}$ can be recovered from the encoding.*

Proof. If P contains some vertex of a fundamental cycle separator used in processing s , let v be the last vertex of P that belongs to the earliest such separator. By choice of the earliest separator, the x -to- v distance (in G_s) is stored (S -to-separator distance). By choice of the last vertex on P that belongs to that separator, the v -to- y distance (in the region of s that contains $P[v, y]$) is stored (separator-to-boundary distance). Thus, the length of P can be recovered. If P contains no such vertex, then x and y are in the same region when the recursion terminates, so the x -to- y distance (in G_s) is stored as a S -to-boundary distance. \square

We extend the previous lemma and show that it applies also to distinguished vertices enclosed by holes of s (i.e., for S^s instead of S_s).

LEMMA 3.5. *The length of a shortest path P in G_s from any $x \in S^s$ to any $y \in C_{ext}$ can be recovered from the encoding.*

Proof. The proof is by induction on the nesting depth of slice s . The base case follows from Lemma 3.4. For the inductive step, if $x \in S_s$ we are done by Lemma 3.4, so assume x is enclosed by some hole h of s .

If P contains some vertex of a fundamental cycle separator used in processing s before hole h is eliminated, let v be the last vertex of P that belongs to the earliest such separator. By choice of the earliest separator, the x -to- v distance (in G_s) is stored (S -to-separator

distance), and by choice of the last vertex of that separator on P , the v -to- y distance (in a region of s that contains $P[v, y]$) is stored (separator-to-boundary distance). Thus, the length of P can be recovered.

If P contains no such vertex, then the artificial vertex v_h and y are in the same region R when either the recursion terminates, or the hole h is eliminated. In either case, the C_h -to- y distances are stored (hole-to-boundary distance). Decompose P into a maximal prefix $P[x, v]$ enclosed by the slice s' whose boundary is C_h , a maximal suffix $P[w, y]$ enclosed by R , and an infix $P[v, w]$. The length of the prefix is stored by the inductive hypothesis for s' . The length of the infix is represented by the boundary-to-boundary distances for s' . The length of the suffix is stored (hole-to-boundary distance). \square

Finally, we extend the previous lemma and show that it applies to any two distinguished vertices.

LEMMA 3.6. *The length of a shortest path P in G_s from any $x \in S^s$ to any $y \in S^s$ can be recovered from the encoding.*

Proof. Assume, wlog, that both x and y are enclosed in holes of s (the other cases are similar and less general). If P contains some vertex of a fundamental cycle separator used in processing s before either hole is eliminated, then let v be a vertex of P that belongs to the earliest such separator. By choice of the earliest separator, both the x -to- v and the y -to- v distances (in G_s) are stored (S -to-separator distance). Thus, the length of P can be recovered.

Otherwise, the hole of x and the hole of y are in the same region R when one of them, say the hole h of x , is eliminated. If P intersects one of the fundamental cycle separators used during the elimination process of

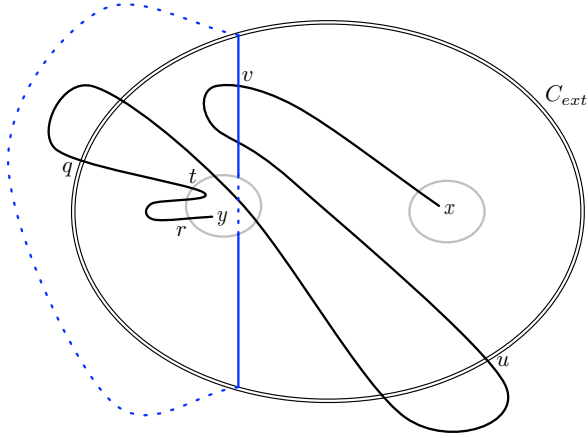


Figure 7: An illustration of one of the cases in the proof of Lemma 3.6. A slice s with two holes is shown. The boundary C_{ext} is double-lined. A x -to- y shortest path P is shown in solid black. The vertices x and y belong to different holes of s (black cycles). The path P crosses a fundamental cycle separator (blue, parts that do not belong to s are dashed) used in eliminating the hole to which y belongs.

hole h , then let v be the last vertex on the earliest such separator (see Figure 7). By choice of earliest separator, the x -to- v distance is stored (S -to-separator distance). By Lemma 3.5, the length of the maximal suffix $P[v, y]$ enclosed in h is also stored. Let u be the first vertex of $P[v, w]$ that belongs to either C_{ext} or C_h (u exists because y is enclosed by C_h and v is not).

- If u belongs to C_{ext} then the length of $P[v, u]$ is stored (separator-to-boundary distance). In this case, let q be the last vertex of P that belongs to C_{ext} . The length of $P[u, q]$ is represented by boundary-to-boundary distances for C_{ext} . Let t be the first vertex of $P[q, y]$ that belongs to C_h . The length of $P[q, t]$ is represented as a hole-to-boundary distance (when h is eliminated). Let r be the last vertex of $P[t, y]$ that belongs to C_h . The length of $P[t, r]$ is represented by boundary-to-boundary distances for C_h , and the length of $P[r, y]$ is represented by Lemma 3.5. See Figure 7 for an illustration.
- If u belongs to C_h then the length of $P[v, u]$ is represented as a separator-to-hole distance. The representation of the suffix $P[u, y]$ is then similar to the previous case.

Finally, we need to treat the case where P does not intersect any fundamental cycle used in eliminating

the hole h . In this case P can be decomposed into a x -to- C_{ext} prefix, a C_h -to- y suffix, and subpaths of P between vertices of $C_h \cup C_{ext}$. The prefix and suffix are represented by Lemma 3.5. The other subpaths are represented as hole-to-boundary or boundary-to-boundary distances as in the two cases above. \square

Finally, we now show that the entire encoding requires only $\tilde{O}(\sqrt{k \cdot n})$ bits.

The encoding size. The space required for the boundary-to-boundary distances for all slices is $O(n/w \cdot \log^2 n)$ since the total boundary size is $O(n/w)$, and by Lemma 2.2.

We next bound the total space required for S -to-separator distances for all slices. Whenever a distinguished vertex stores its distances to a path P explicitly, the total weight of its region decreases by a constant factor within $O(\log n)$ recursive steps (either immediately, if this happens in the outer recursion, or otherwise by the time the hole-elimination process ends). So this can happen $O(\log n)$ times per distinguished vertex. Because $|P| = O(w)$ (by the height of T_s), this sums up to a total of $O(k \cdot w \cdot \log^3 n)$ bits.

Consider now a specific slice s . We have already argued that the depth of the recursive process to handle a slice is $O(\log^2 n)$. Similarly to the analysis in Section 3.1 of the single hole case, the total space required for storing separator-to-boundary distances using Lemma 2.1 or Lemma 2.3 at all calls at the same recursive level is $O((|\partial s| + (k_s - 1) + w(k_s - 1)) \log n)$, where ∂s denotes the boundary of slice s , and k_s denotes the number of distinguished nodes in slice s plus the number of holes of s that enclose at least one distinguished node. Note that k_s is an upper bound on the number of regions at any level of the recursive process handling s . Also note that a slice with $k_s = 1$ is not partitioned at all, so stores no separator-to-boundary distances. Hence the terms with $(k_s - 1)$. For exactly the same reasons, the total space required for storing separator-to-hole distances using Lemma 2.1 at all calls at the same recursive level (this only happens in the hole-elimination recursion) is $O((|\partial_h s| + (k_s - 1) + w(k_s - 1)) \log n)$, where $\partial_h s$ denotes the boundaries of the holes of s . Summing over all $O(\log^2 n)$ recursive levels, the space required for storing separator-to-boundary and separator-to-hole distances for slice s is $O((|\partial s| + |\partial_h s| + (k_s - 1) + w(k_s - 1)) \log^3 n)$. By Lemma 3.1, $\sum \partial s + \partial_h s$ over all slices is $O(n/w)$. We claim that, since the total number of distinguished nodes in G is k , the sum $\sum (k_s - 1)$ over all slices is $O(k)$. Assuming the claim, the total space required for storing separator-to-boundary and separator-to-hole distances for all slices is $O((n/w + k + wk) \log^3 n)$.

To prove the claim consider a tree whose nodes represent slices s with $k_s \geq 1$, or distinguished vertices.

The root of the tree is the slice whose external boundary is the infinite face of G . The descendants of a node representing a slice s are the nodes representing the distinguished nodes in s as well as the nodes representing slices s' with $k_{s'} \geq 1$ whose external boundaries are the holes of s . Thus, the degree of the node representing a slice s is k_s . The total number of leaves of this tree is k , and the sum represents the sum over $\deg(v) - 1$ over all nodes v in the tree. This sum is bounded by the number of leaves in the tree, which is k , proving the claim.

Hole-to-boundary distances are stored using Lemma 2.3 for at most one hole in each region along the recursion. Each invocation of Lemma 2.3 for hole C_h and boundary fragment b_i requires $O((|C_h| + |b_i|) \log n)$ bits. For a single level of the recursion, this sums up to $O(((k_s - 1) + |\partial s| + |\partial_h s|) \log n)$ because there are at most $2(k_s - 1)$ vertices that contribute in more than one region (endpoints of b_i 's). As above, summing over all levels of the recursion and all slices, this gives $O((k + n/w) \log^3 n)$. The bound for S -to-boundary distances is $O((k + n/w) \log n)$ for the same reasons.

To conclude, we showed that the total size of the entire encoding is bounded by $O((n/w + k \cdot w) \log^3 n)$, which is $O(\sqrt{k \cdot n} \cdot \log^3 n)$ by choosing $w = \sqrt{n/k}$.

4 A Tight Lower Bound

Recall that Gavaille et al. [38] show how to construct, given a Boolean $\frac{k}{2} \times \frac{k}{2}$ matrix B , a planar grid $G(B)$ containing $O(k^3)$ vertices, such that B can be recovered from the distances between k distinguished vertices of $G(B)$. This shows that, for $k \leq n^{1/3}$, encoding all distances between k vertices of a planar graph requires $\Theta(k^2)$ bits. For $k \geq n^{1/3}$, we consider t Boolean $\frac{k}{2t} \times \frac{k}{2t}$ matrices B_1, B_2, \dots, B_t . For each of these matrices, we construct a planar grid containing $O((\frac{k}{2t})^3)$ vertices. The disjoint union of all these grids is a planar graph on $O(t(\frac{k}{2t})^3) = O(k^3/t^2)$ vertices, such that all Boolean matrices can be recovered from the distances between the k distinguished vertices. Hence, encoding all such distances requires $\Omega(t(\frac{k}{2t})^2) = \Omega(k^2/t)$ bits. Setting $t = \sqrt{k^3/n}$ we obtain that encoding all distances between the k distinguished vertices of a planar graph on n vertices requires $\Omega(\sqrt{k \cdot n})$ bits.

5 Query Time

The goal of Section 3 was to guarantee that all distances between distinguished vertices are captured, but we were not concerned with the complexity of retrieving such a distance. In this section we explain how to augment the encoding to allow efficient extraction of the stored distances, proving Theorem 1.2.

We start with reformulating our encoding using the

notion of *dense distance graphs*. Vertices of a dense distance graph are listed explicitly, but its edges are described implicitly with unit Monge matrices. Each such matrix describes lengths of the edges between every $u \in U$ and $v \in V$, for some subsets of nodes U and V . The matrix is represented using $O(|U| + |V|)$ words as described in Lemma 2.1. In particular, we may have $|U| = |V| = 1$ and then the matrix simply stores the length of a single edge explicitly. The size of a dense distance graph is the total number of vertices plus the sum of $|U| + |V|$ over all matrices describing length of the edges. By construction, our encoding described in Section 3 is based on defining a dense distance graph of size $O(\sqrt{k \cdot n} \log^2 n)$. Note that in Section 3 we measure size by bits, and here by size of the dense distance graph. Every distinguished node of the original graph is a vertex of the dense distance graph, and the distance between two distinguished nodes of the original (unweighted) graph is the same as the distance between their corresponding vertices in the (weighted) dense distance graph. Fakcharoenphol and Rao designed an efficient algorithm for computing the shortest paths in such a graph, nicknamed the FR-Dijkstra:⁵

LEMMA 5.1. ([35]) *Distance between any two vertices of a dense distance graph of size s can be found in $O(s \log^2 s)$ time.*

Applying Lemma 5.1 gives us an oracle of size $O(\sqrt{k \cdot n} \log^2 n)$ words, answering queries in $\tilde{O}(\sqrt{k \cdot n} \log^4 n)$ time. For very large k , say $k = \Omega(n)$, the query time is clearly not optimal, as there exists an oracle of size $\tilde{O}(n)$ answering queries in $\tilde{O}(\sqrt{n})$ [35] time. In the remaining part of this section we will describe how to construct an oracle of size $\tilde{O}(\sqrt{k \cdot n})$ answering queries in $\tilde{O}(n^{3/4})$ time.

To improve the query time, we apply the vanilla planar separator lemma.

LEMMA 5.2. *For any planar graph G on n nodes, there exists a partition of the nodes of G into sets A , B , and S , such that $|A|, |B| \leq \frac{2}{3}n$, $|S| = O(\sqrt{n})$, and there are no edges between the nodes of A and B .*

We recursively apply Lemma 5.2 to construct a hierarchical decomposition of the whole graph. The recursion is described by a binary tree \mathcal{K} , where every node $u \in \mathcal{K}$ corresponds to an induced subgraph $G(u)$

⁵In the original paper and most of the subsequent work, the dense distance graph is obtained from an r -division of a planar graph. The vertices are the boundary nodes and distances between boundary nodes in the same region are represented with multiple Monge matrices. However, it is easy to see that their algorithm work for any dense distance graph as defined above.

of the original graph. We let $n(u)$ and $s(u)$ denote the number of nodes and distinguished nodes in $G(u)$, respectively. We terminate the recursion as soon as $s(u) \leq \sqrt{n(u)}$. If $u \in \mathcal{K}$ is a leaf, we define its set of distinguished nodes $D(u)$ to consist of all the distinguished nodes of $G(u)$. Otherwise, $D(u)$ consists of the following nodes:

1. the separator of $G(u)$,
2. for every child v of u that is a leaf, all the distinguished nodes of $G(v)$,
3. for every child v of u that is not a leaf, the separator of $G(v)$.

Then, we construct a dense distance graph of size $O(\sqrt{n(u)|D(u)|} \log^2 n(u))$ capturing distances between any two nodes from $D(u)$ in $G(u)$.

To calculate the distance $d_G(u, v)$ between two distinguished nodes u and v in G , we locate the deepest nodes u' and v' of \mathcal{K} , such that $u \in G(u')$ and $v \in G(v')$. Then, we consider the union of all dense distance graphs constructed for the nodes of \mathcal{K} on the paths from u' and v' to the root. Note that the same node of G might appear in more than one of these dense distance graphs, and we identify all of its copies. By construction, the obtained dense distance graph captures the sought distance. Furthermore, its size is bounded by

$$\begin{aligned} & \max_{u \in \mathcal{K}} O(\sqrt{n(u)|D(u)|} \log^2 n(u)) = \\ & \max_{u \in \mathcal{K}} O(\sqrt{n(u)\sqrt{n(u)}} \log^2 n(u)) = O(n^{3/4} \log^2 n). \end{aligned}$$

Therefore, by Lemma 5.1 we can answer a query in $O(n^{3/4} \log^4 n)$ time. It remains to bound the size of the resulting oracle.

LEMMA 5.3. *The dense distance graph constructed for node $u \in \mathcal{K}$ is of size $O(\sqrt{n(u) \cdot s(u)} \cdot \log^2 n(u))$.*

Proof. To prove the lemma it is enough to bound $|D(u)|$ by $O(s(u))$. If u is a leaf, this is clear. Otherwise, $s(u) > \sqrt{n(u)}$ and $D(u)$ consists of the following nodes:

1. the separator of $G(u)$ of size $O(\sqrt{n(u)}) = O(s(u))$.
2. for every child v of u that is a leaf, all $s(v) \leq s(u)$ distinguished nodes of $G(v)$,
3. for every child v of u that is not a leaf, the separator of $G(v)$ of size $O(\sqrt{n(v)}) = O(\sqrt{n(u)}) = O(s(u))$.

Node u has at most two children, so indeed $|D(u)| = O(s(u))$. \square

To upper bound the size of the oracle, we need to upper bound the sum $\sum_{u \in \mathcal{K}} \sqrt{n(u) \cdot s(u)} \cdot \log^2 n(u)$. To this end, we separately consider all nodes $u \in \mathcal{K}$ such that $n(u) \in [(\frac{2}{3})^{\ell+1}n, (\frac{2}{3})^\ell n)$, for every $\ell = 0, 1, \dots, O(\log n)$. Fix ℓ and call these nodes u_1, u_2, \dots, u_t . Then, no u_i is a descendant of another u_j , so every node of the original graph appears in at most one $G(u_i)$. Therefore, $\sum_i s(u_i) \leq k$ and $\sum_i n(u_i) \leq n$. From the latter inequality and the lower bound on $n(u)$ we obtain that $t \leq (\frac{3}{2})^{\ell+1}$. Now we want to upper bound the following sum:

$$\begin{aligned} \sum_i \sqrt{n(u_i) \cdot s(u_i)} & \leq \sqrt{\left(\frac{2}{3}\right)^\ell n} \sum_i \sqrt{s(u_i)} \\ & = O(\sqrt{n/t} \cdot \sum_i \sqrt{s(u_i)}). \end{aligned}$$

From the concavity of $f(x) = \sqrt{x}$, the above sum is maximized when $s(u_i) = k/t$, so we obtain:

$$\sum_i \sqrt{n(u_i) \cdot s(u_i)} = O(\sqrt{n/t} \cdot t \cdot \sqrt{k/t}) = O(\sqrt{k \cdot n}).$$

To obtain an upper bound on $\sum_{u \in \mathcal{K}} \sqrt{n(u) \cdot s(u)} \cdot \log^2 n(u)$, we only need to multiply the above bound by $\log n$ because for every $u \in \mathcal{K}$ there exists ℓ such that $n(u)$ belongs to the appropriate interval, so the total size of the oracle is $O(\sqrt{k \cdot n} \cdot \log^3 n)$.

6 Labeling Schemes for Unit-Monge Matrices

A distance labeling scheme is a way to compress graphs that allows for distributed decoding. The goal is to assign a label ℓ_v for each node v , so that by looking at the labels of two nodes ℓ_s, ℓ_t (without access to the original graph) we can infer the distance between them $d(s, t)$. The main question one asks about such schemes is *how small can the labels be?* A famous open question is to close the gap between the $O(\sqrt{n})$ upper bound [38, 42] and the $\Omega(n^{1/3})$ [38] lower bound for planar graphs. The only known technique capable of proving a tight lower bound is via a lower bound for the metric compression problem: if you show that the metric cannot be compressed into $O(k \cdot n^{1/2-\epsilon})$ bits, then you show that no labels of size $O(n^{1/2-\epsilon})$ are possible. Our work deems this approach impassable, since such compressions are indeed possible. Optimistically, it is natural to ask if our upper bound for compression could lead to a better upper bound for labeling. Our encoding assigns $o(n^{1/2})$ bits per node, but can we distribute these bits to the nodes while allowing any pair of nodes to deduce the distance from their local information?

Unfortunately, it appears that there is a fundamental difficulty with converting our encoding into a distance labeling scheme. The heart of our encoding is

a method for capturing pairwise distances between a large subset of nodes of the graph using space proportional to the size of the subset, see Lemma 2.1. A key part in the proof of the lemma is an efficient encoding of an $n \times n$ matrix into $\tilde{O}(n)$ bits, as long as it has the *unit-Monge* property. Translating this into a distance labeling scheme would require being able to assign a label to every row and every column of a given unit-Monge matrix M , such that $M[i, j]$ can be computed from the label of the i -th row and the j -th column. We call this labeling unit-Monge matrices for extracting an element. In this section we show almost tight bounds for the size of a label in this problem. The lower bound shown in Lemma 6.2 is a strong indication that converting our structure into labeling scheme with labels of length $\tilde{O}(n^{1/3})$ would require a major modification.

We start with recalling the following connection between unit-Monge matrices and permutation matrices. P is a permutation matrix if every row and every column contains at most one 1 and 0s elsewhere. Then, it is straightforward to verify that, for any permutation matrix P the matrix M defined as $M[i, j] = \sum_{i' \geq i, j' \geq j} P[i', j']$ is a unit-Monge matrix. In fact, essentially any unit-Monge matrix can be obtained through such transformation, see e.g. Section 2 in [74].

LEMMA 6.1. *For any unit-Monge matrix M , there exists a permutation matrix P , such that $M[i, j] = H[i] + V[j] + \sum_{i' \geq i, j' \geq j} P[2i', 2j']$, for some vectors H and V .*

Consequently, we can focus on assigning a label to every row and column of P , such that given the label of the i -th row and the j -th column we can compute $\sum_{i' \geq i, j' \geq j} P[i', j']$. We call this problem labeling $n \times n$ permutation matrices for dominance sum queries.

LEMMA 6.2. *Labeling $n \times n$ permutation matrices for dominance sum queries is possible with $O(\sqrt{n} \log n)$ bits.*

Proof. We can assume that there is exactly one 1 in every row and column of P . Therefore, the input is fully described by a permutation π . Any permutation on n elements can be decomposed by up to \sqrt{n} increasing subsequences I_1, I_2, \dots and up to \sqrt{n} decreasing subsequences D_1, D_2, \dots . The label of every row and every column consists of $O(\log n)$ bits stored for every such subsequence, thus $O(\sqrt{n} \log n)$ bits in total. We think of every subsequence as a set of points $(x_1, y_1), (x_2, y_2), \dots$ and the $O(\log n)$ bits corresponding to this subsequence in the label of the i -th row and the j -th column should be enough to determine the number of points (x_k, y_k) such that $x_k \geq i$ and $y_k \geq j$. We separately describe what should be stored for an increasing subsequence and then for a decreasing subsequence.

Consider an increasing subsequence consisting of points $(x_1, y_1), (x_2, y_2), \dots$, such that $x_k < x_{k+1}$ and $y_k < y_{k+1}$ for every $k = 1, 2, \dots$. Then, the label of the i -th row stores the smallest k such that $x_k \geq i$, and similarly the label of the j -th row stores the smallest k such that $y_k \geq j$. By taking the maximum of these two numbers we can determine the number of points (x_k, y_k) such that $x_k \geq i$ and $y_k \geq j$.

Now consider a decreasing subsequence consisting of points $(x_1, y_1), (x_2, y_2), \dots$, such that $x_k < x_{k+1}$ and $y_k > y_{k+1}$ for every $k = 1, 2, \dots$. Then, the label of the i -th (j -th) row stores the smallest (largest) k such that $x_k \geq i$ ($y_k \geq j$). Denoting the number stored for the i -th row and the j -th row by ℓ and r , respectively, the number of points (x_k, y_k) such that $x_k \geq i$ and $y_k \geq j$ can be calculated as $\max(0, r - \ell + 1)$. \square

LEMMA 6.3. *Labeling $n \times n$ permutation matrices for dominance sum queries requires $\Omega(\sqrt{n})$ bits.*

Proof. We conceptually divide an $n \times n$ matrix P into blocks of size $\sqrt{n} \times \sqrt{n}$, thus creating an $\sqrt{n} \times \sqrt{n}$ matrix B , where every entry $B[i, j]$ corresponds to a block of P . For every block $B[i, j]$ we choose one bit $b[i, j]$. We will show that then it is always possible to construct the matrix P , such that all bits $b[i, j]$ can be retrieved from the labels of rows of the form $1 + \alpha \cdot \sqrt{n}$ and columns of the form $1 + \alpha \cdot \sqrt{n}$. Then it follows that we can encode n bits of information in $2\sqrt{n}$ labels, hence one of these labels must consist of $\frac{1}{2}\sqrt{n}$ bits.

We construct P incrementally. We call a row or a column of P active if there is no 1 there. We start with an empty P and keep adding 1s there while making ensure that there is at most one 1 in every row and column. Given the labels of all rows $1 + \alpha \cdot \sqrt{n}$ and all columns of the form $1 + \alpha \cdot \sqrt{n}$ we can count 1s in every block of P . The goal is to ensure that this count is equal to $b[i, j]$. Assume that this is already the case for every $b[i, j]$ such that $i < i'$ or $i = i'$ and $j < j'$ and consider $b[i', j']$. If $b[i', j'] = 0$ we continue. Otherwise, we have to choose exactly one active row r in the range $[1 + i' \cdot \sqrt{n}, (i' + 1) \cdot \sqrt{n}]$ and exactly one active column c in the range $[1 + j' \cdot \sqrt{n}, (j' + 1) \cdot \sqrt{n}]$, and set $P[r, c] = 1$, thus making both r and c inactive. This clearly guarantees that there is exactly one 1 in the corresponding block of P . The only problem is to guarantee that there is at least one active row and column in the appropriate ranges. However, we have deactivated less than i' rows in the range $[1 + i' \cdot \sqrt{n}, (i' + 1) \cdot \sqrt{n}]$ so far, and similarly less than j' columns in the range $[1 + j' \cdot \sqrt{n}, (j' + 1) \cdot \sqrt{n}]$, so indeed there exist active row and column. \square

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