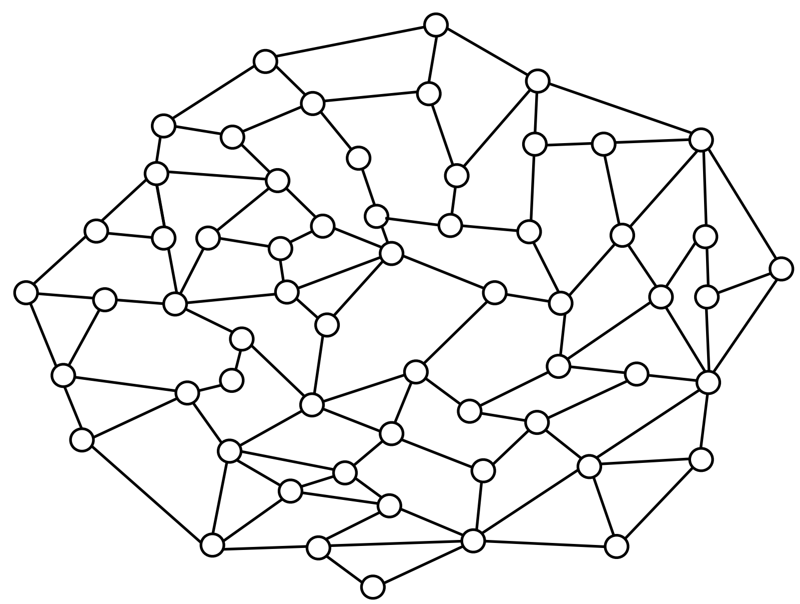
Computing the Girth of a Planar Graph in $O(n \log n)$ time



Oren Weimann (Weizmann Institute of Science)

Raphy Yuster (University of Haifa)

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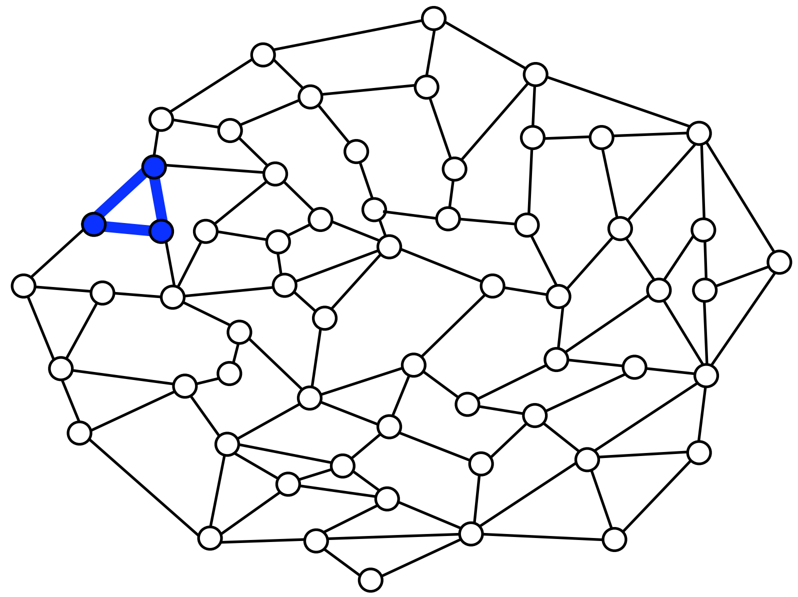

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Shortest Cycle (Girth)

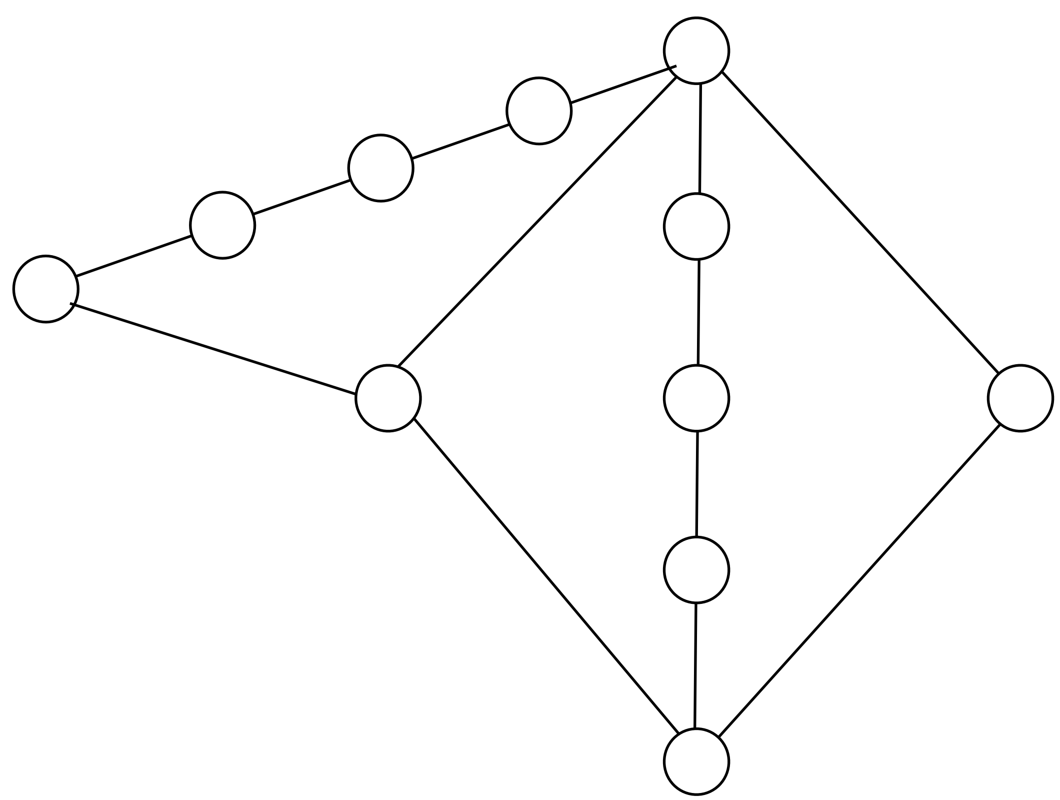
General graphs:

- $O(nm)$ - [Itai and Rodeh 1978]
- $O(n^2\alpha(\mathrm{n}))$ - [Monien1983] only even length
- $O(n^2)$ - [Yuster and Zwick 1997] only even length

Planar graphs:

- $O(n)$ - [Papadimitriou and Yannakakis 1981] girth bounded by 3
- $O(n)$ - [Eppstein 1999] girth bounded by any constant
- $O(n^{5/4} \log n)$ - [Djidjev 2000]
- $O(n \log^2 n)$ - [Chalermsook, Fakcharoenphol, Nanongkai 2004]

Girth = Min-Cut in dual



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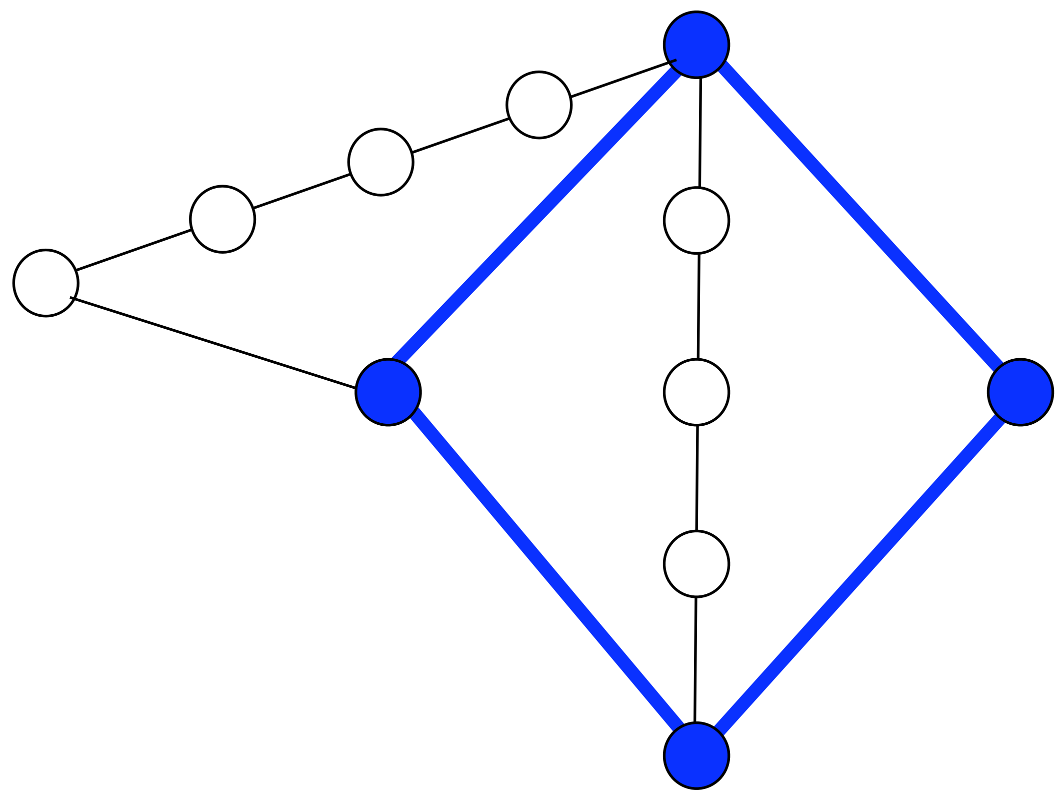

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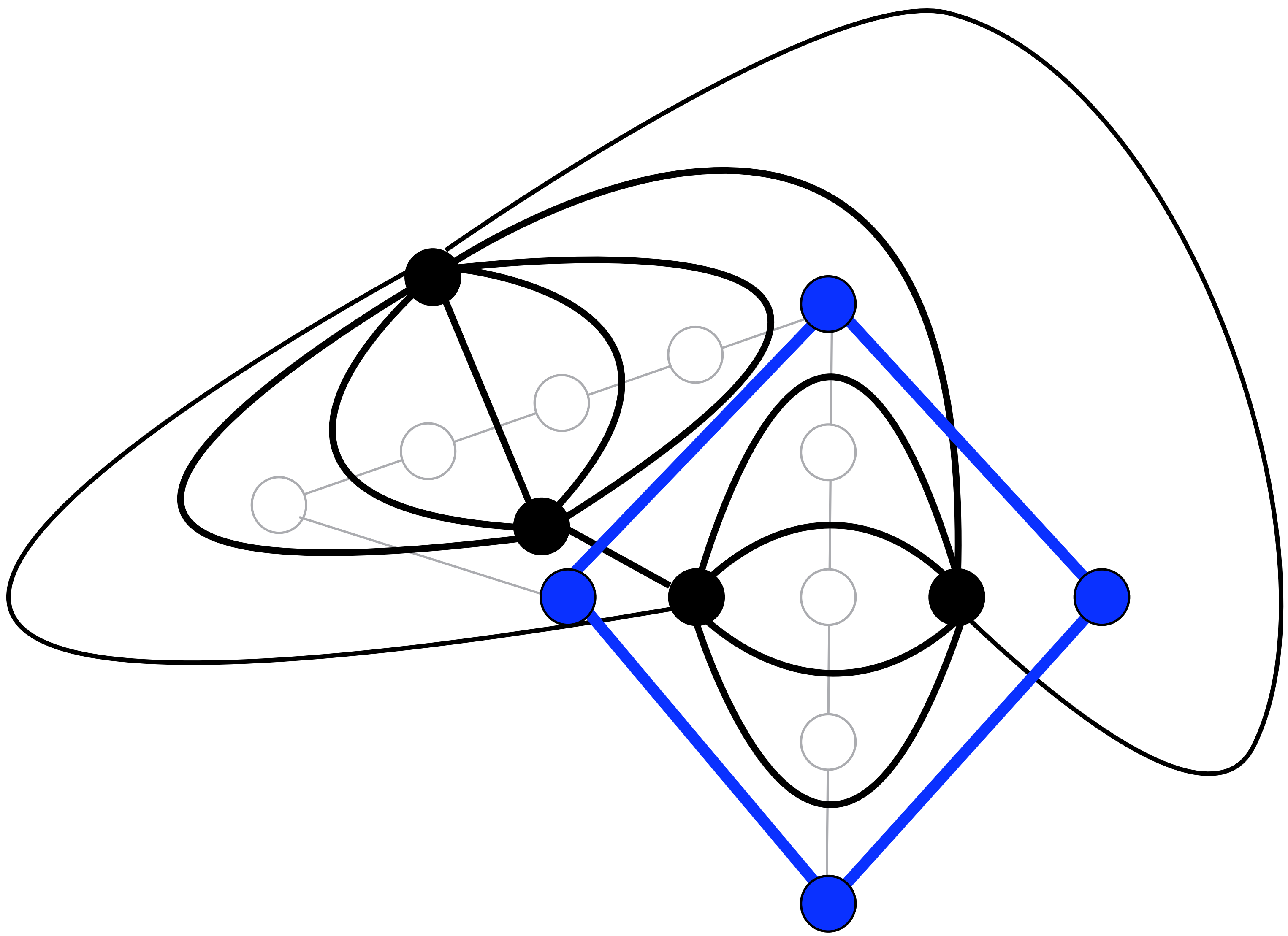

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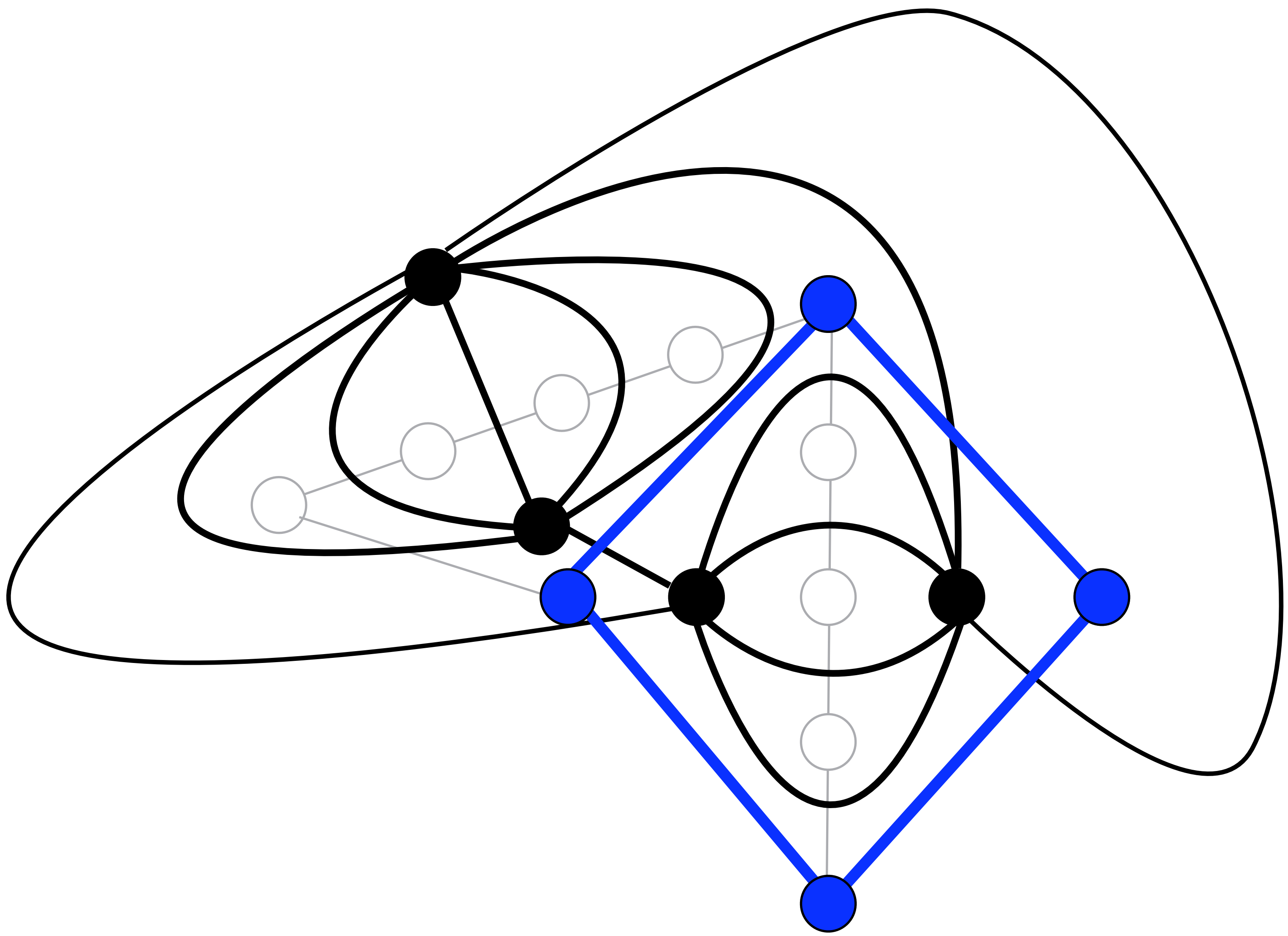

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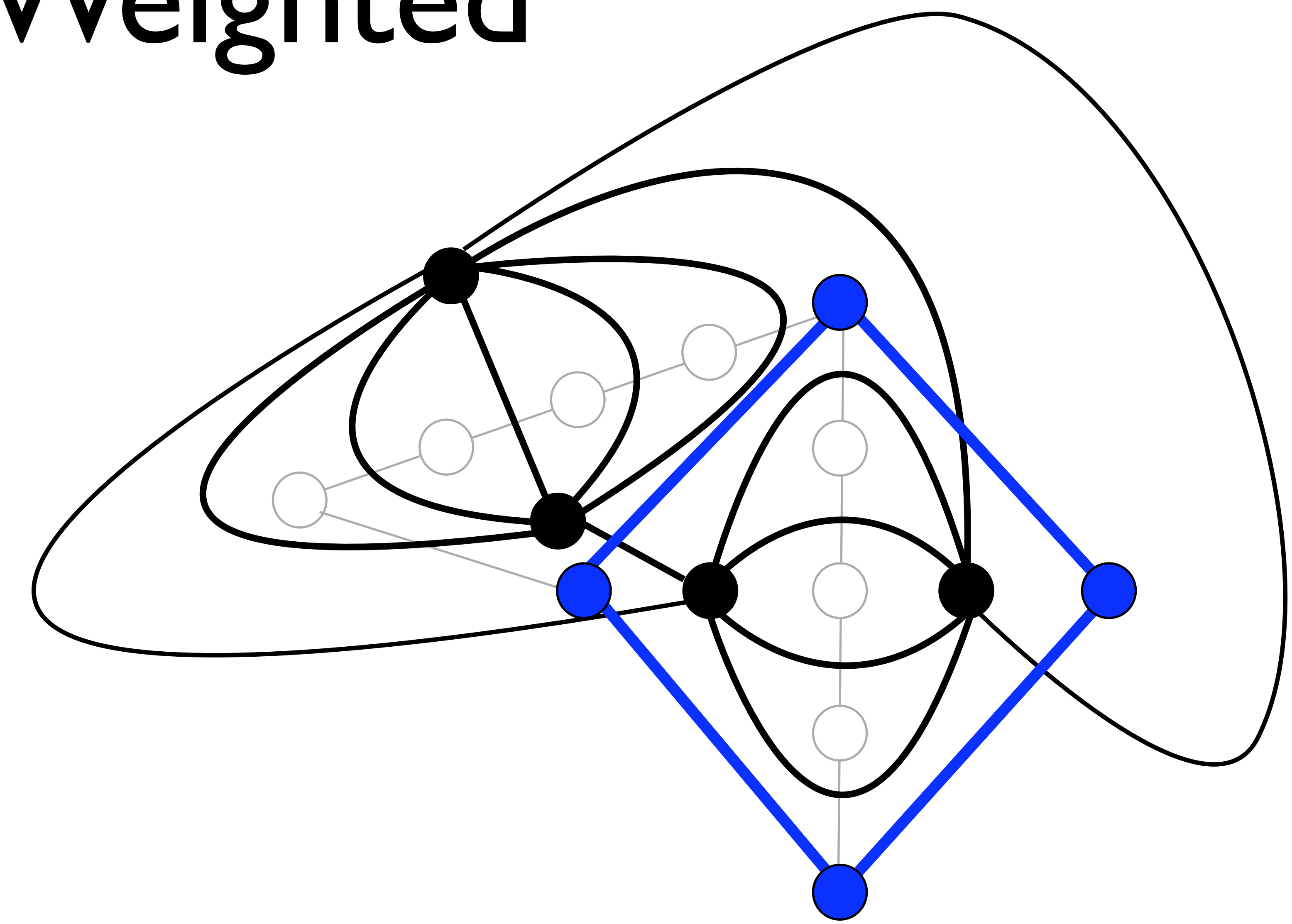
Weighted



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Our Contribution:

- $O(n \log n)$ - bounded genus, don't need embedding, simple

Shortest Cycle in Planar Graphs

- Undirected
- Unweighted

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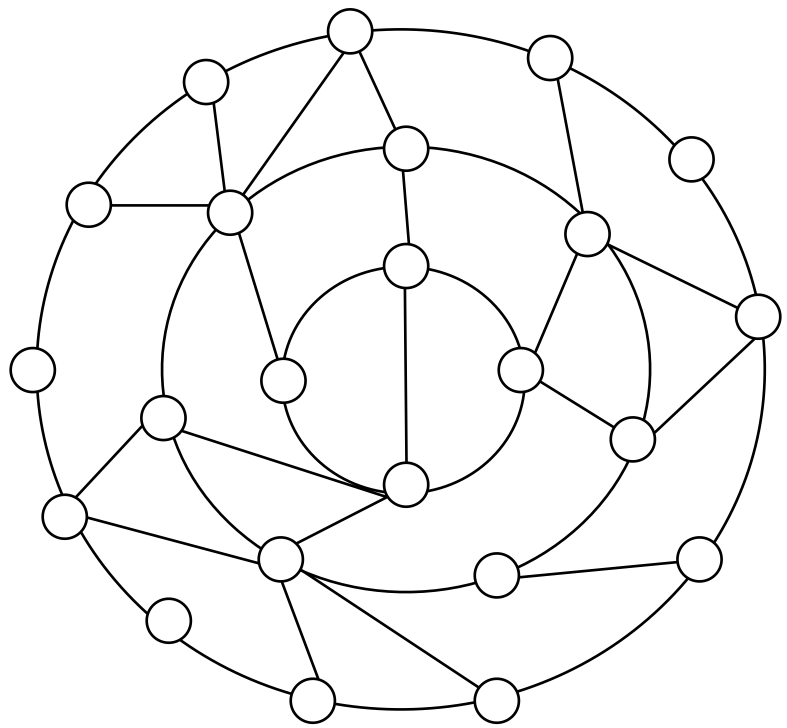

k-outerplanar graphs: $O(kn\log n)$

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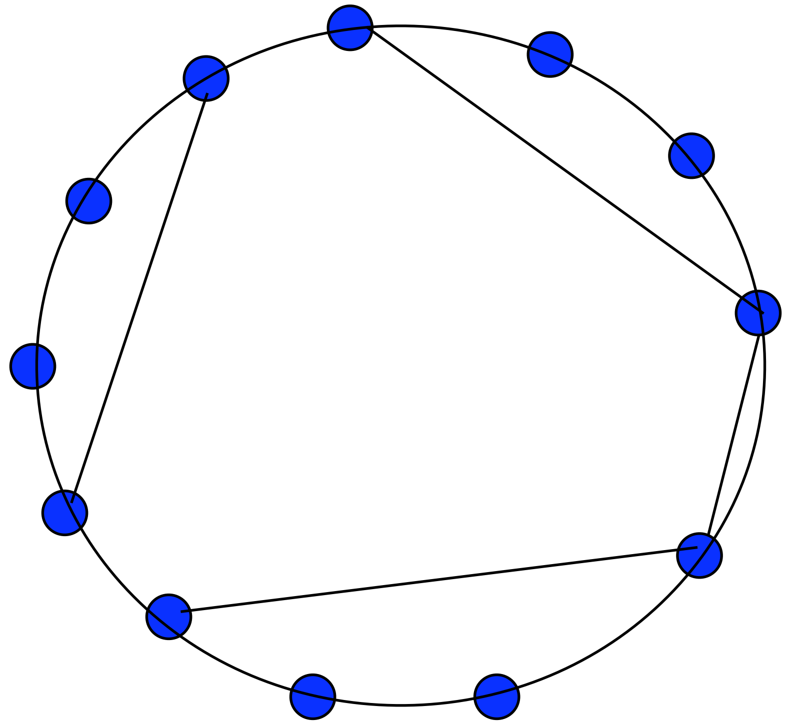

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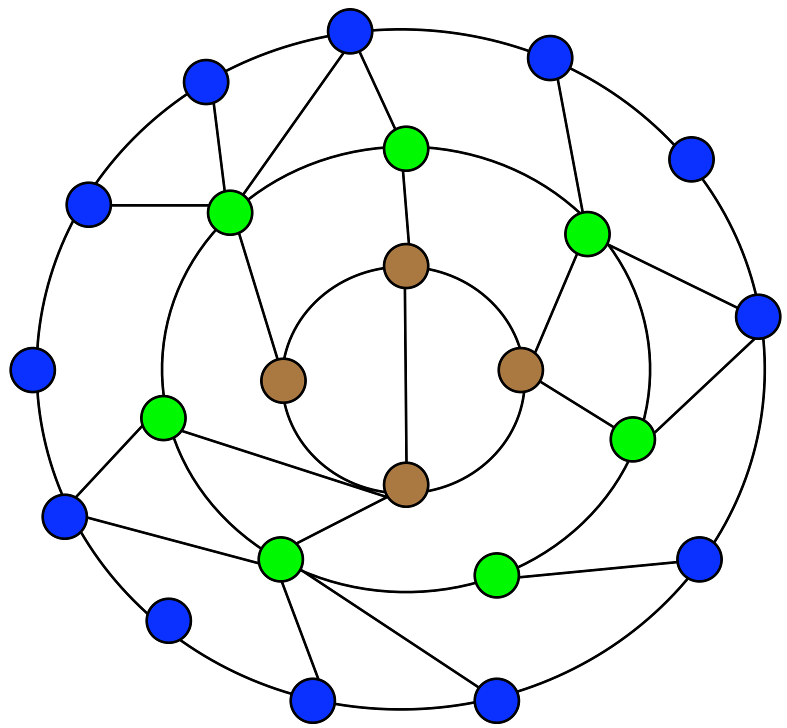

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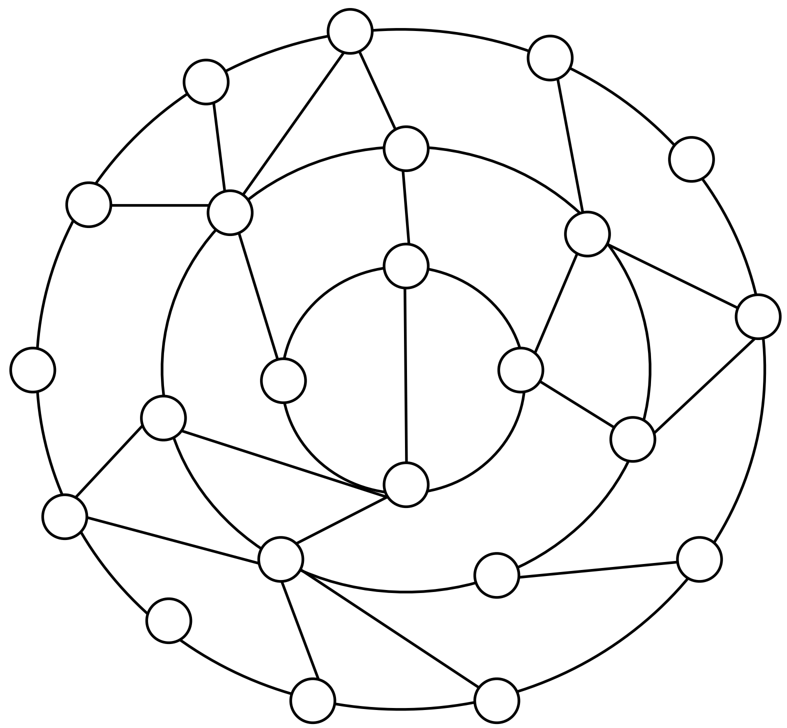

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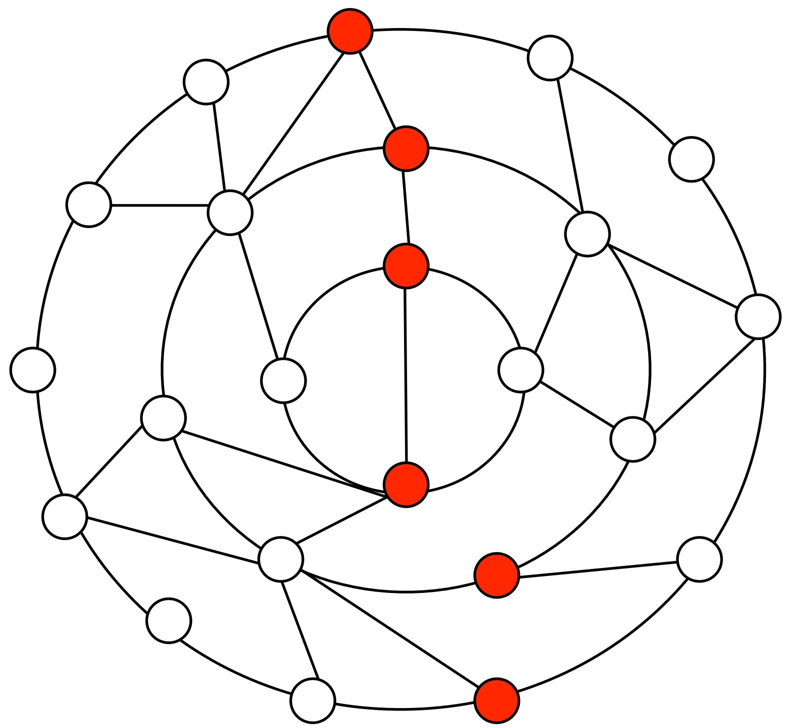

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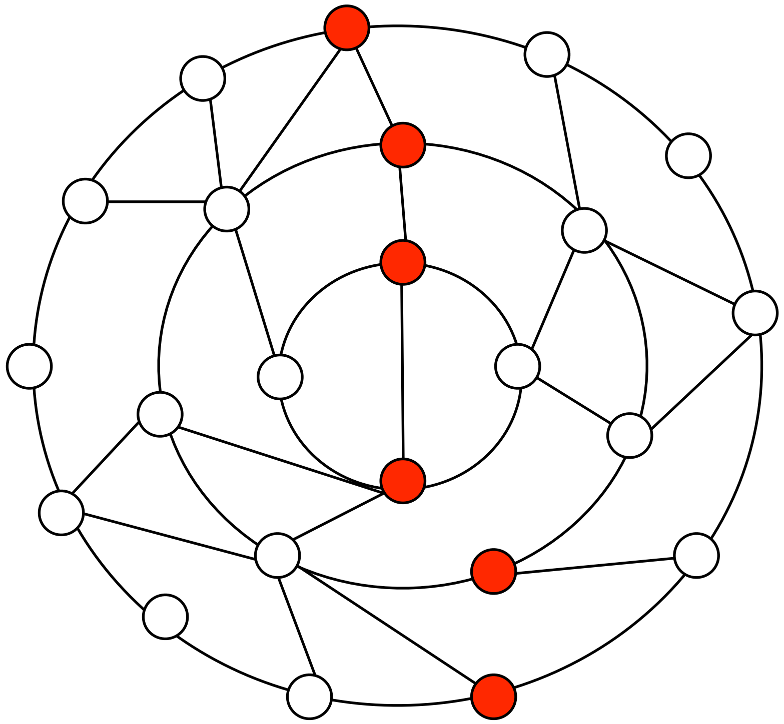

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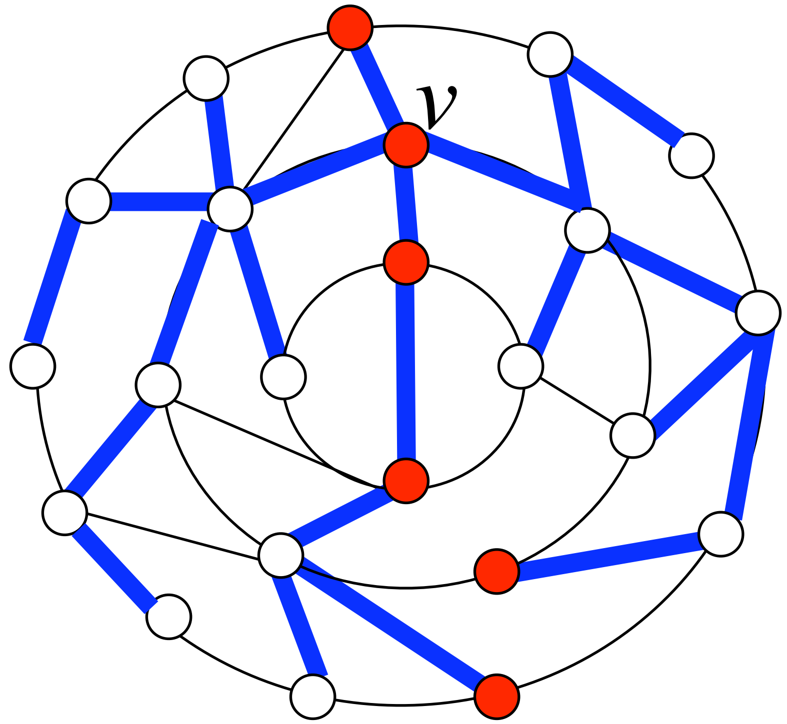

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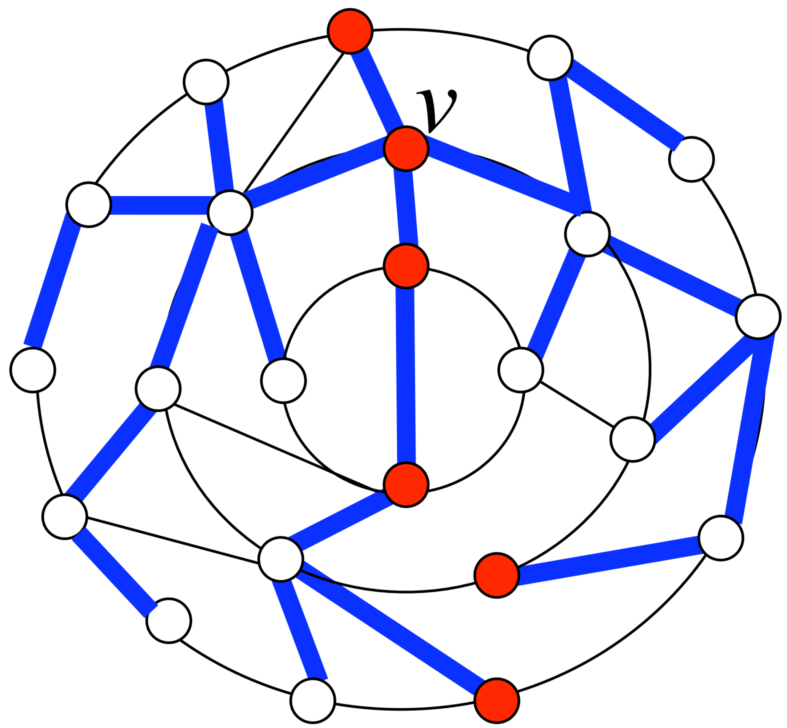

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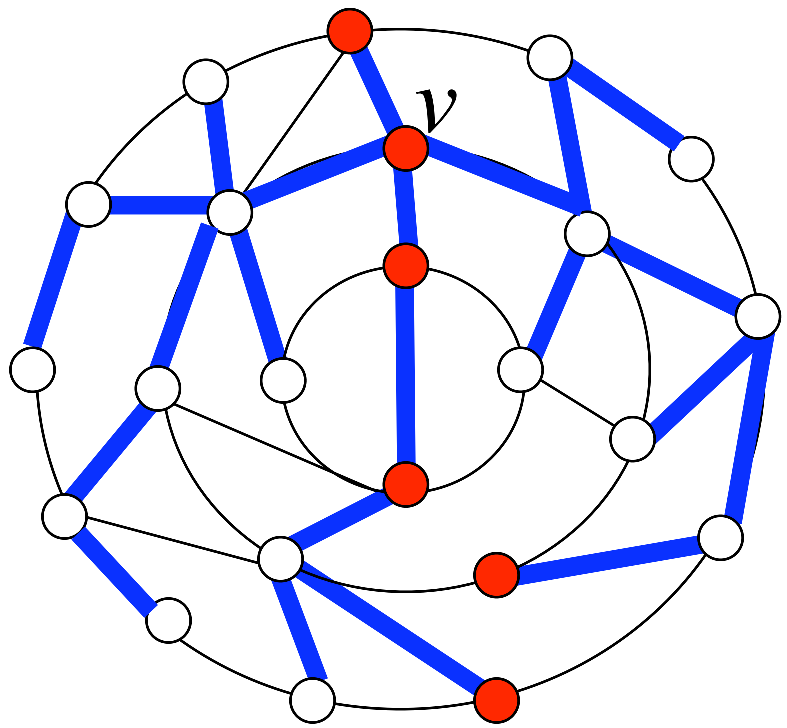

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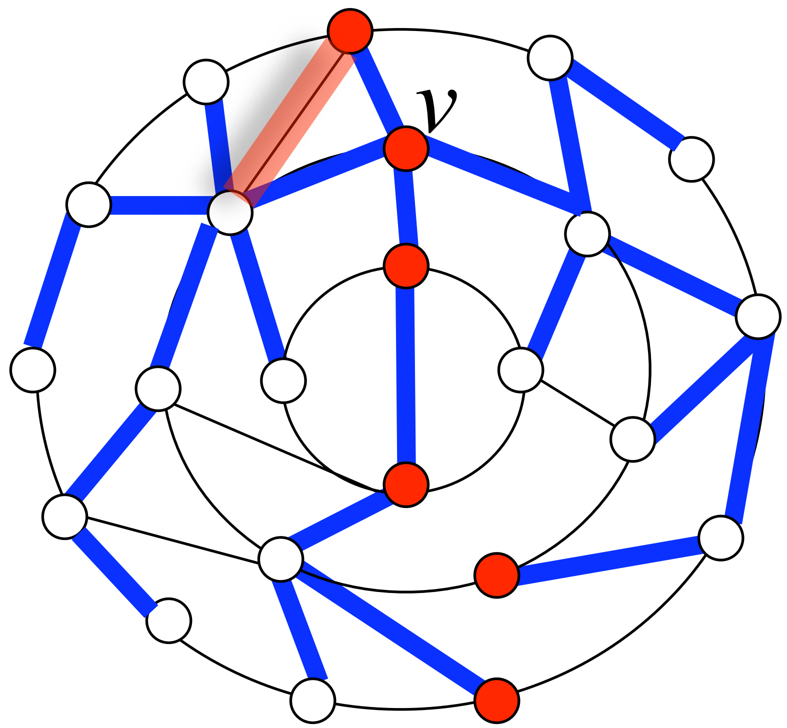

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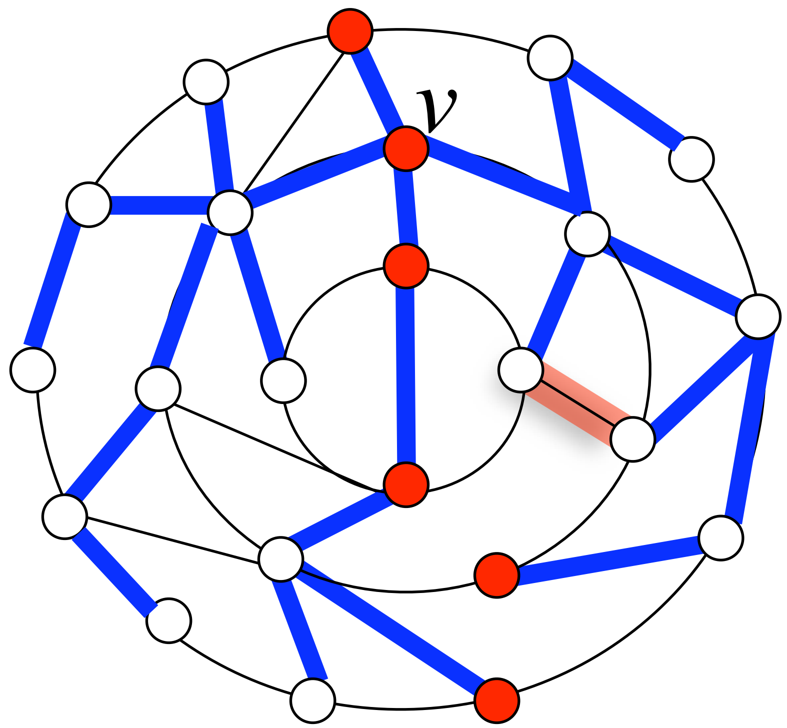

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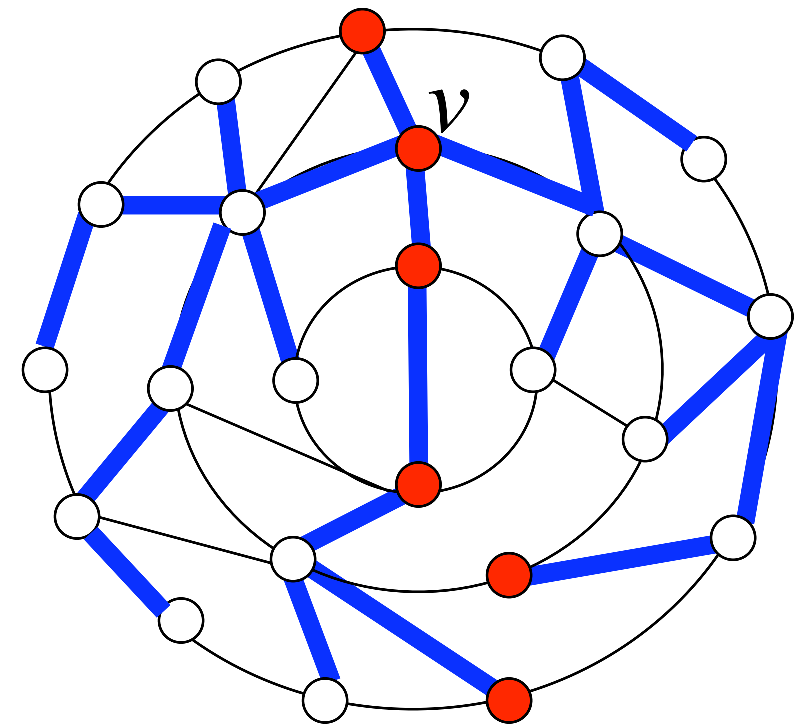

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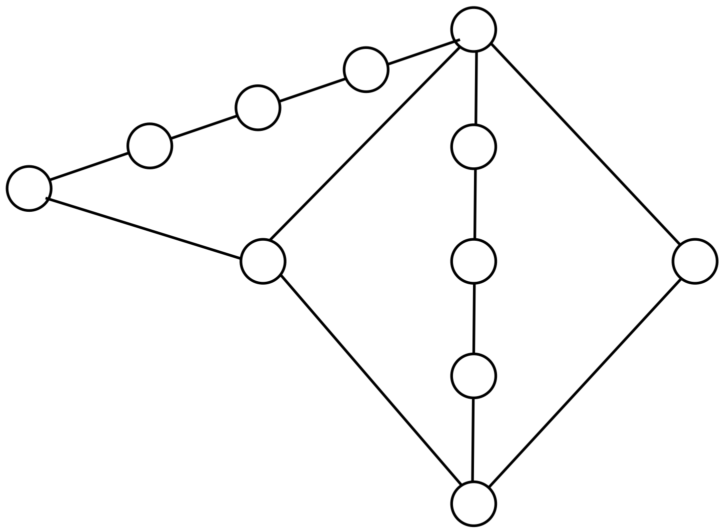
- *good*: 1. Works even for weighted graphs
  2. $O(n^{3/2})$ for any planar graph, even directed
- *bad*: Does not necessarily find shortest simple cycle through v

Some Simple Observations



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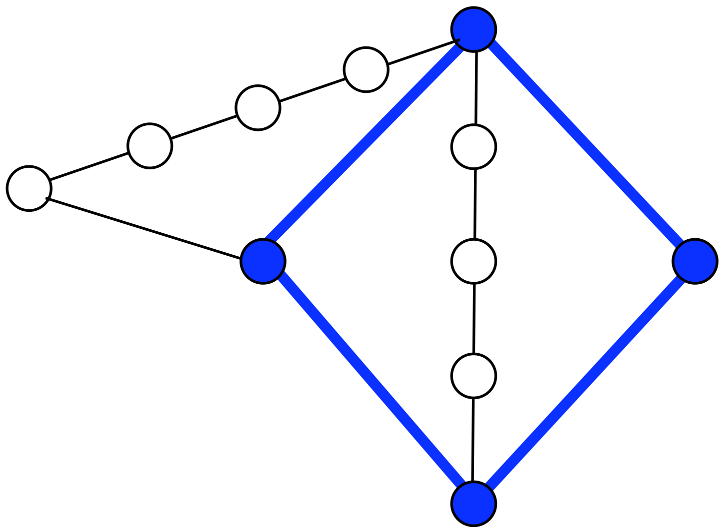
- Girth g



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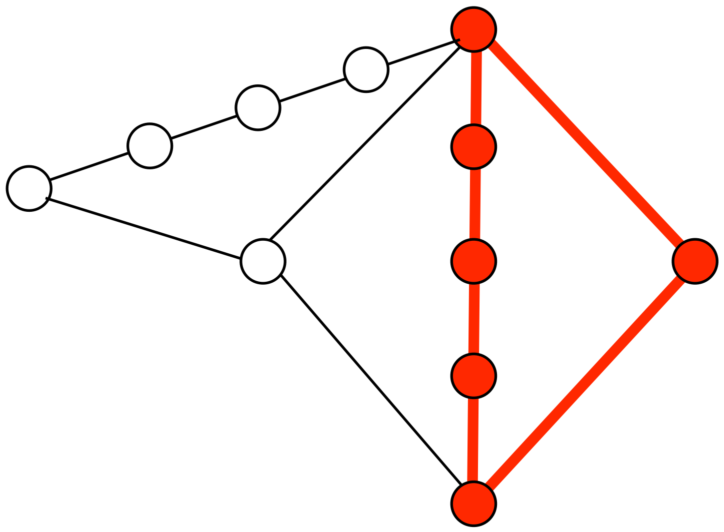

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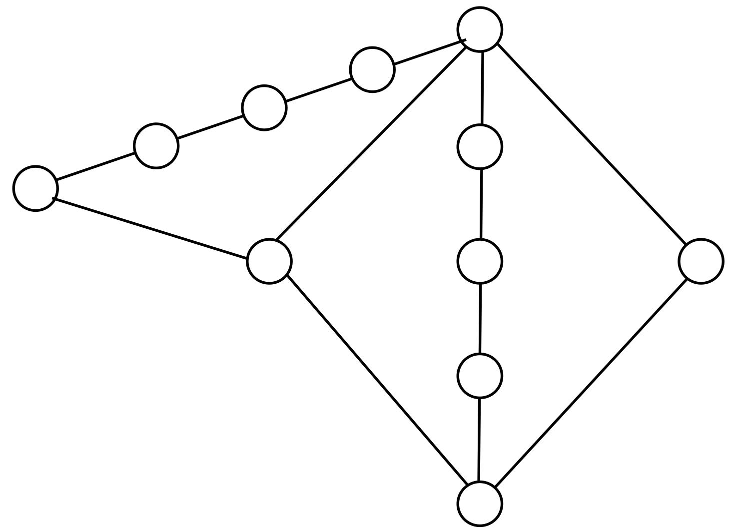

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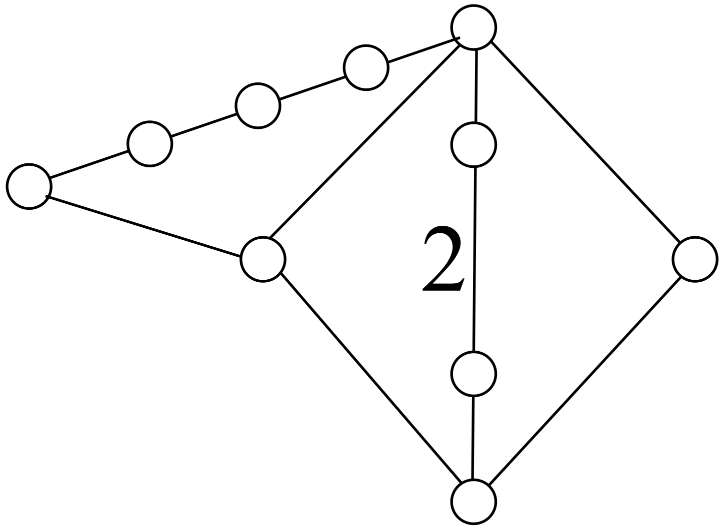

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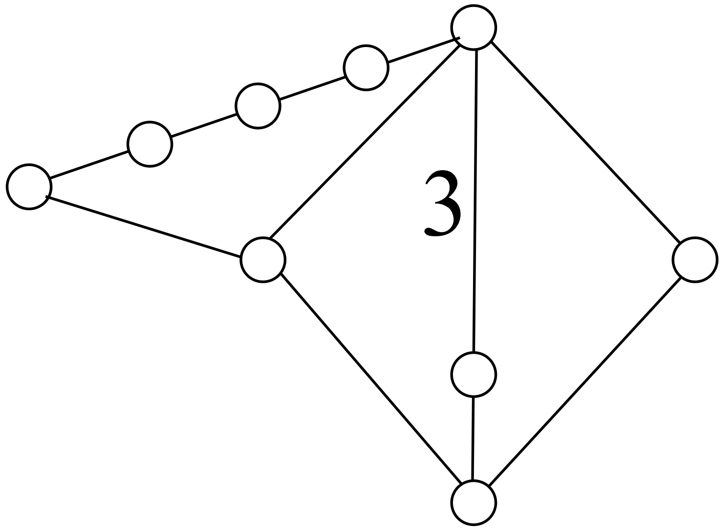

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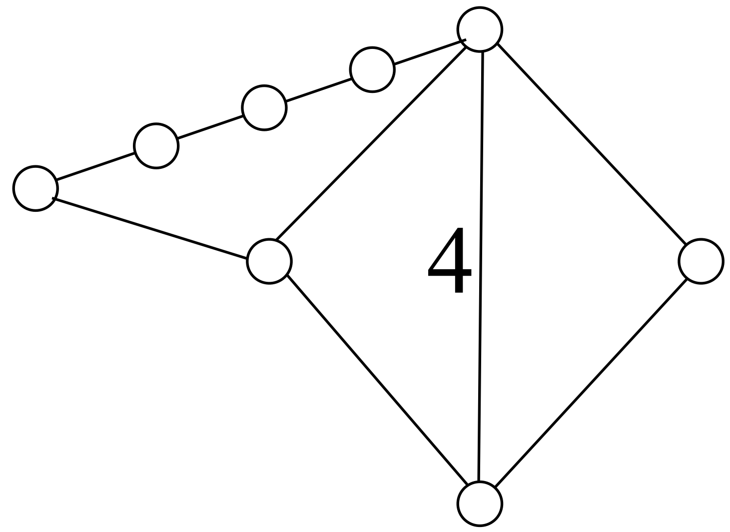

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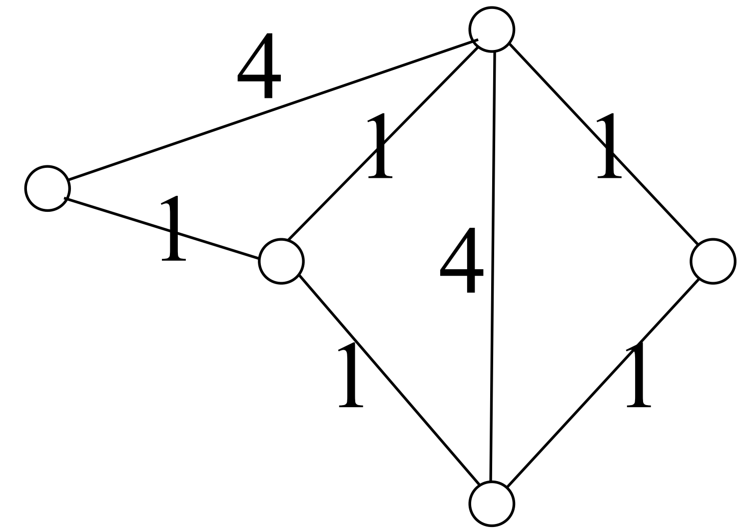

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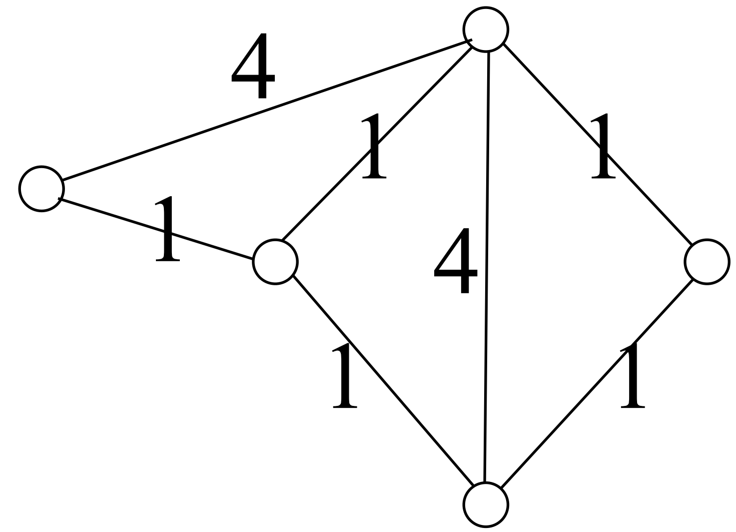

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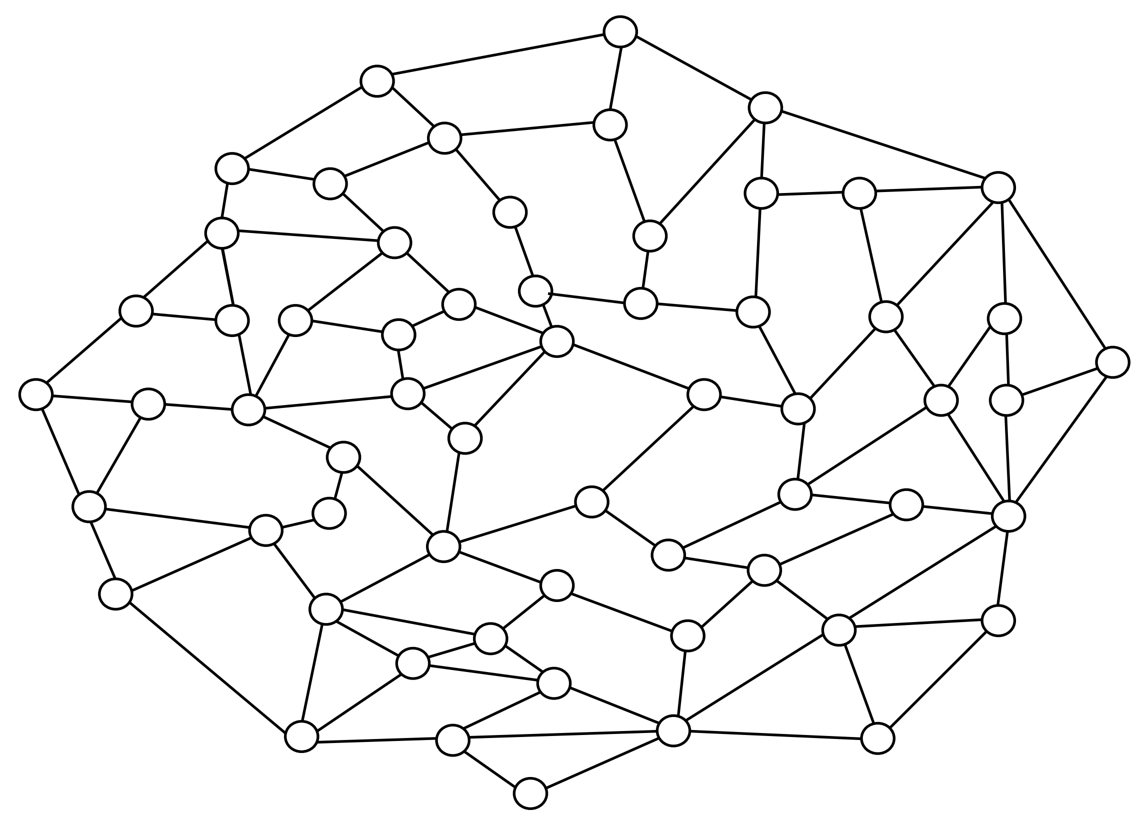
- *proof sketch:*
  - Every edge belongs to two faces, and every face has at least h vertices
  - We can remove vertices of degree 0 or 1
  - We can assume G' is not a simple cycle
    $\Rightarrow$ degree 2 vertices form an independent-set
  - $\sum$ degree(v) $= 2m' = \Theta(n')$
  - Euler's formula: $m' = n' + f' - 2$

The Algorithm

$|G'| = n/h$



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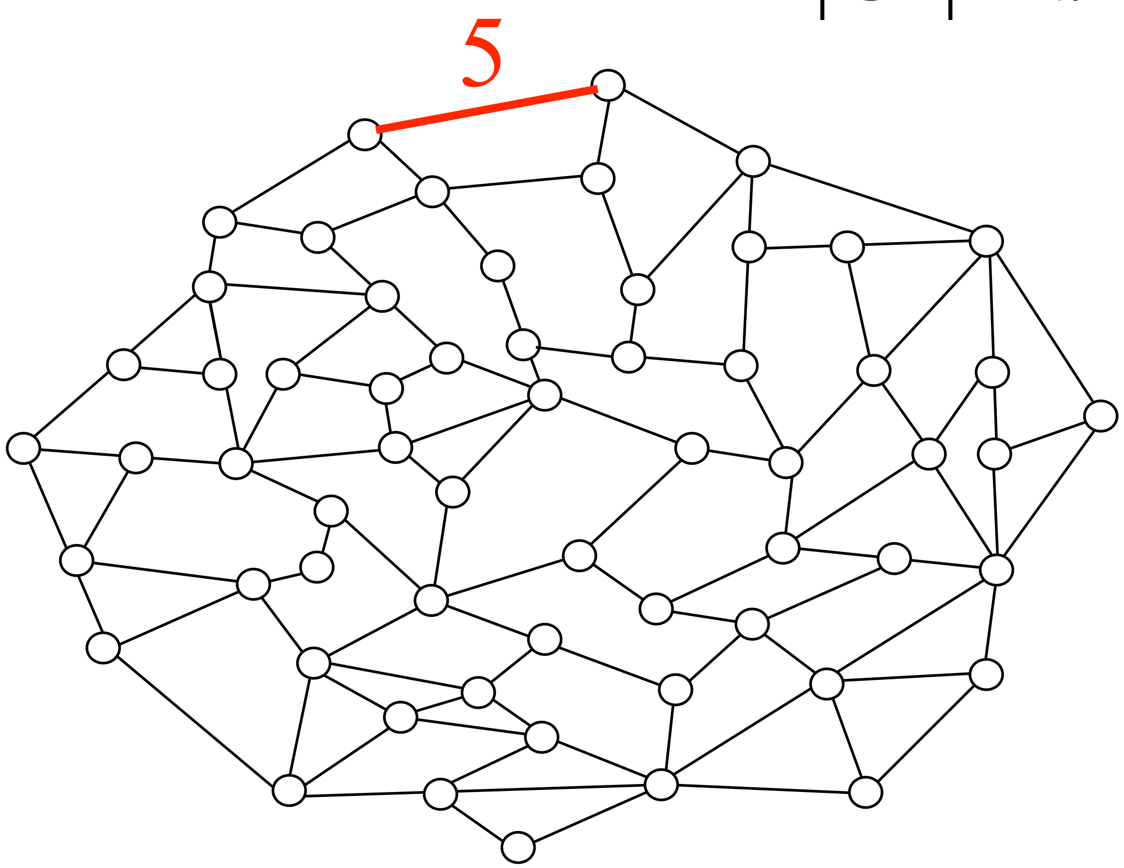

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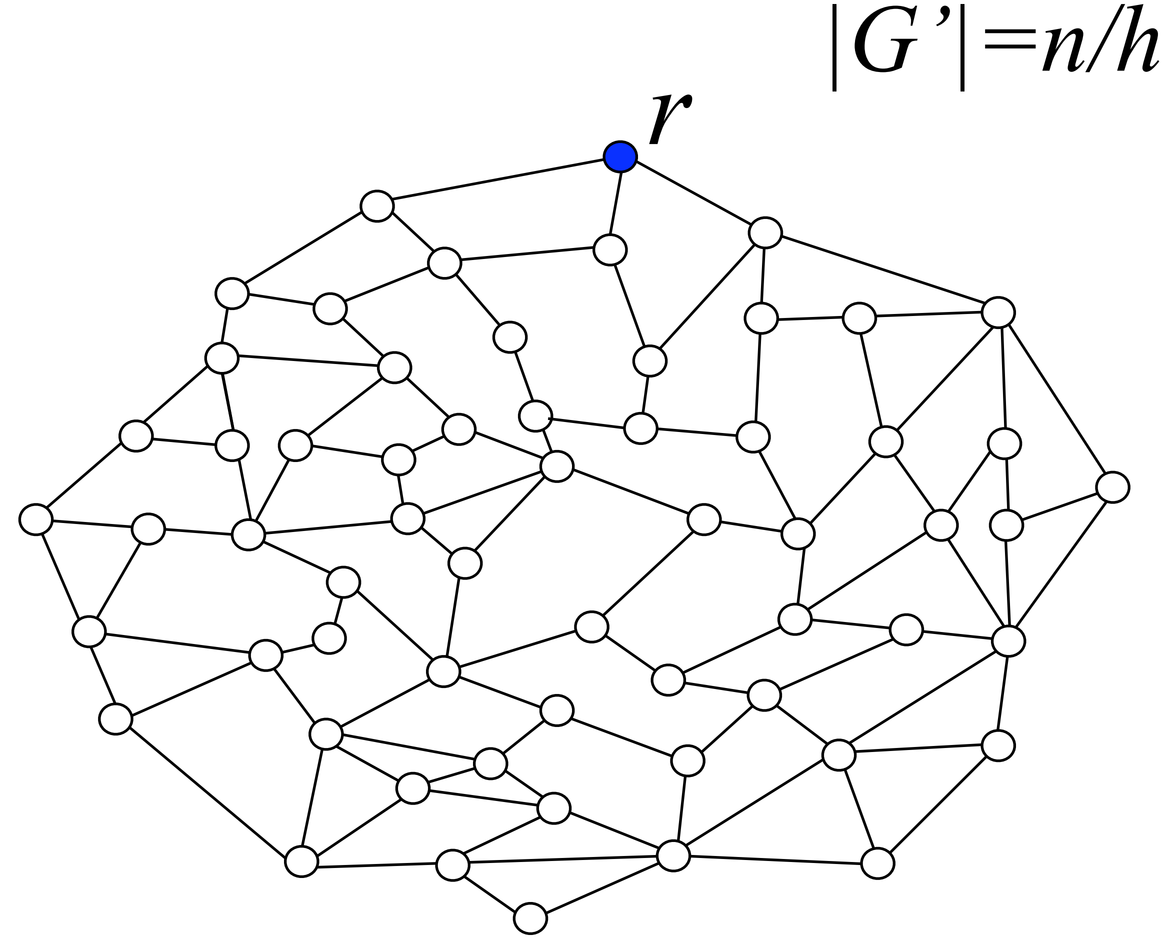

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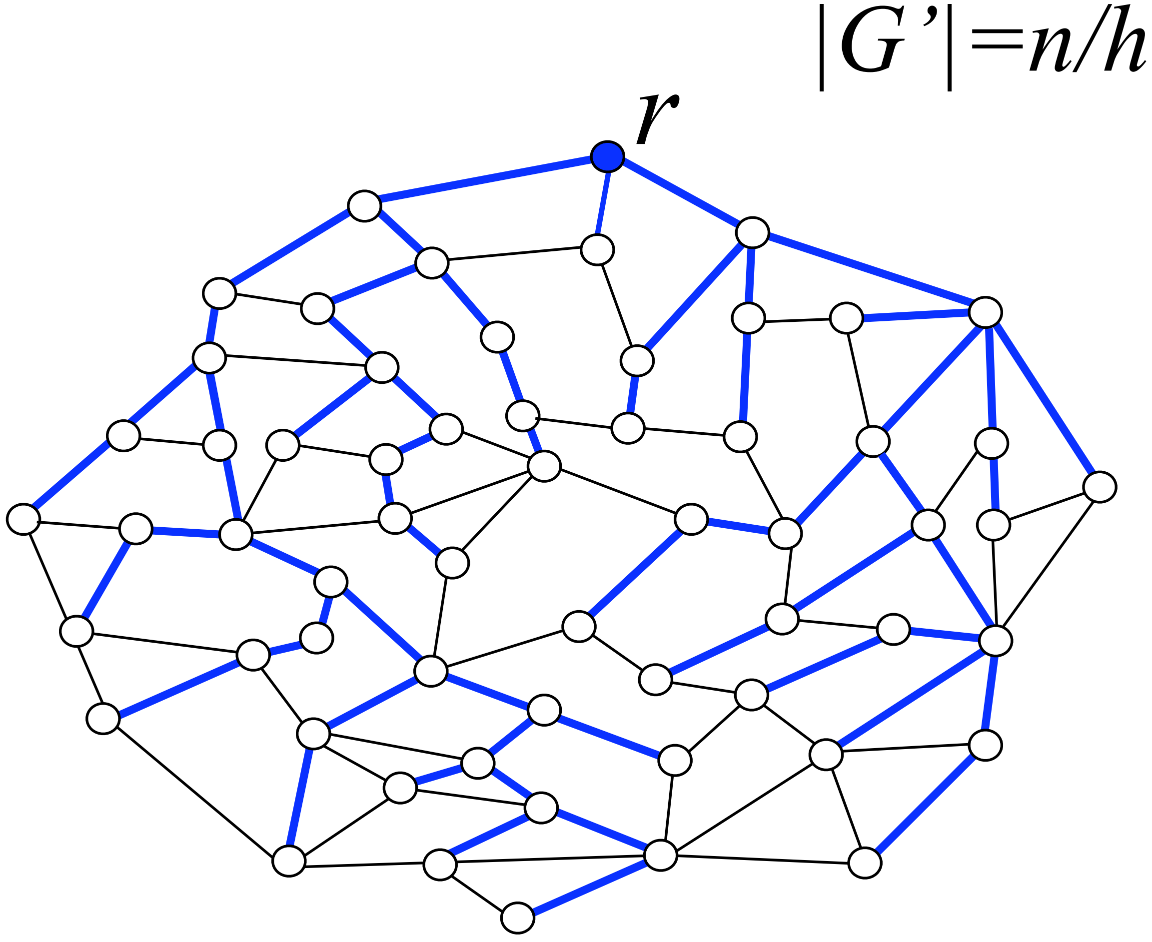

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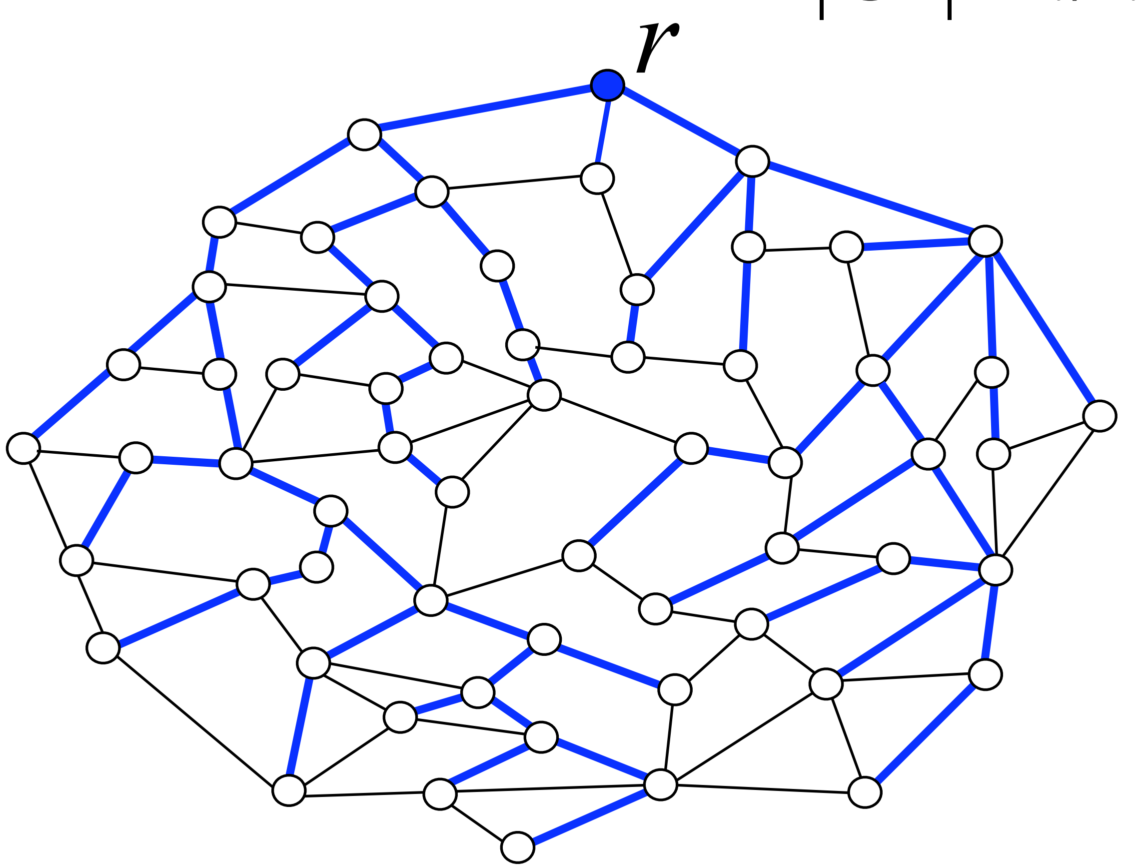

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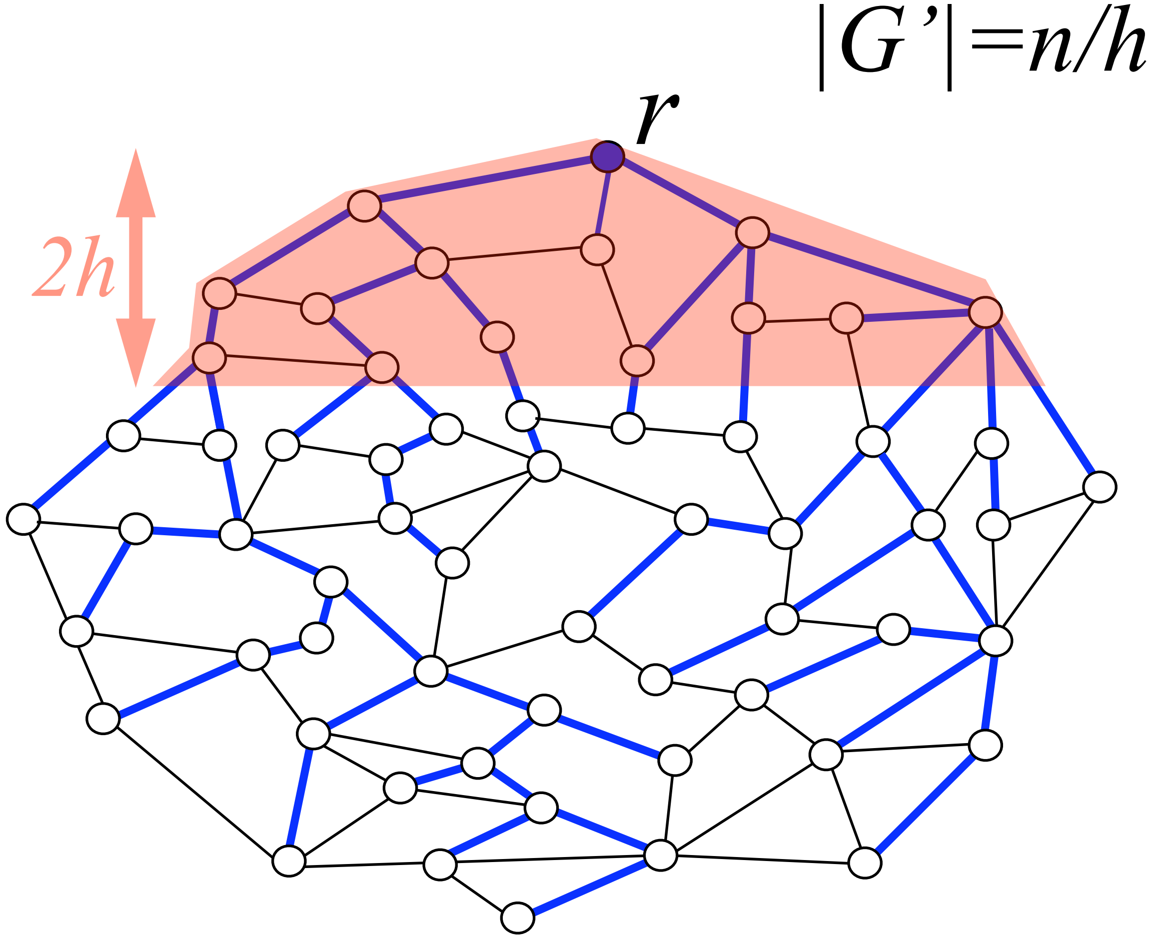

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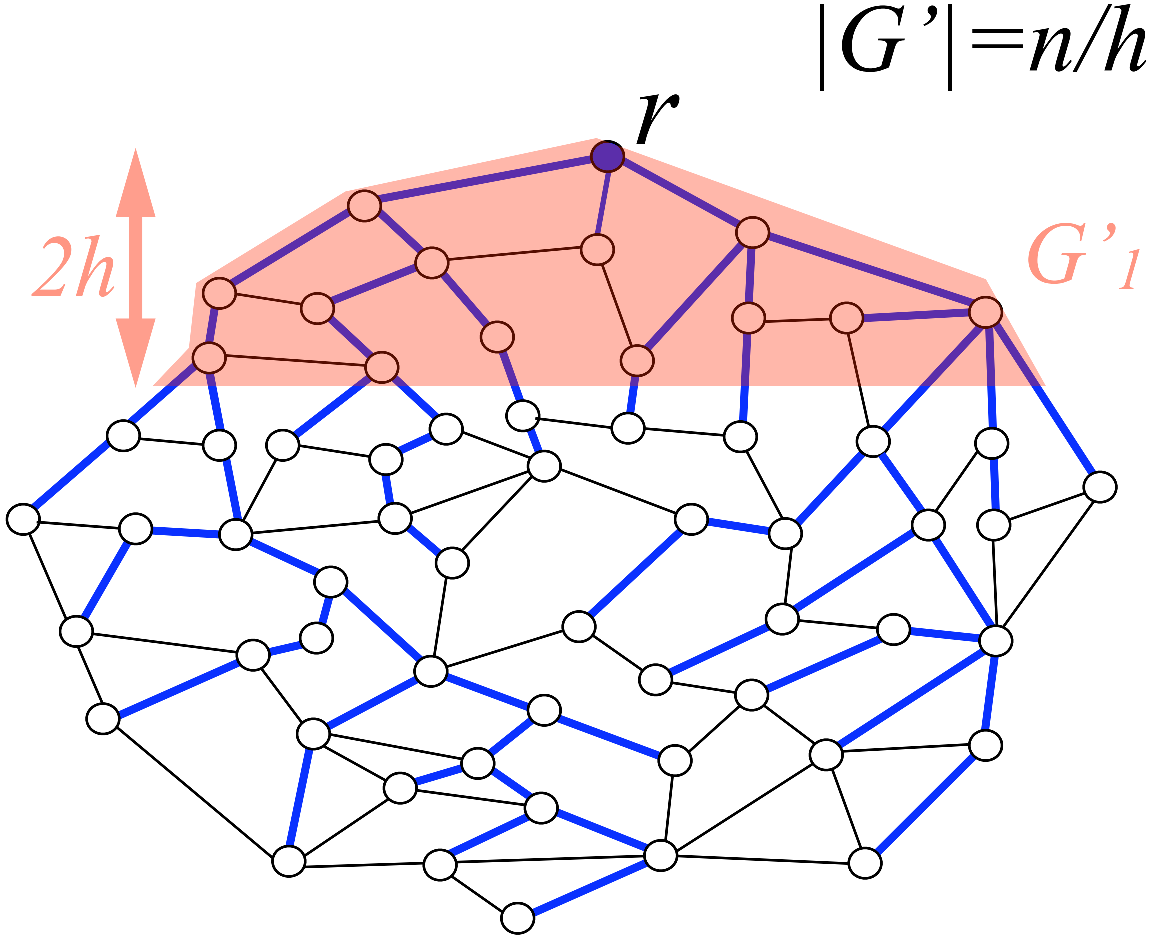

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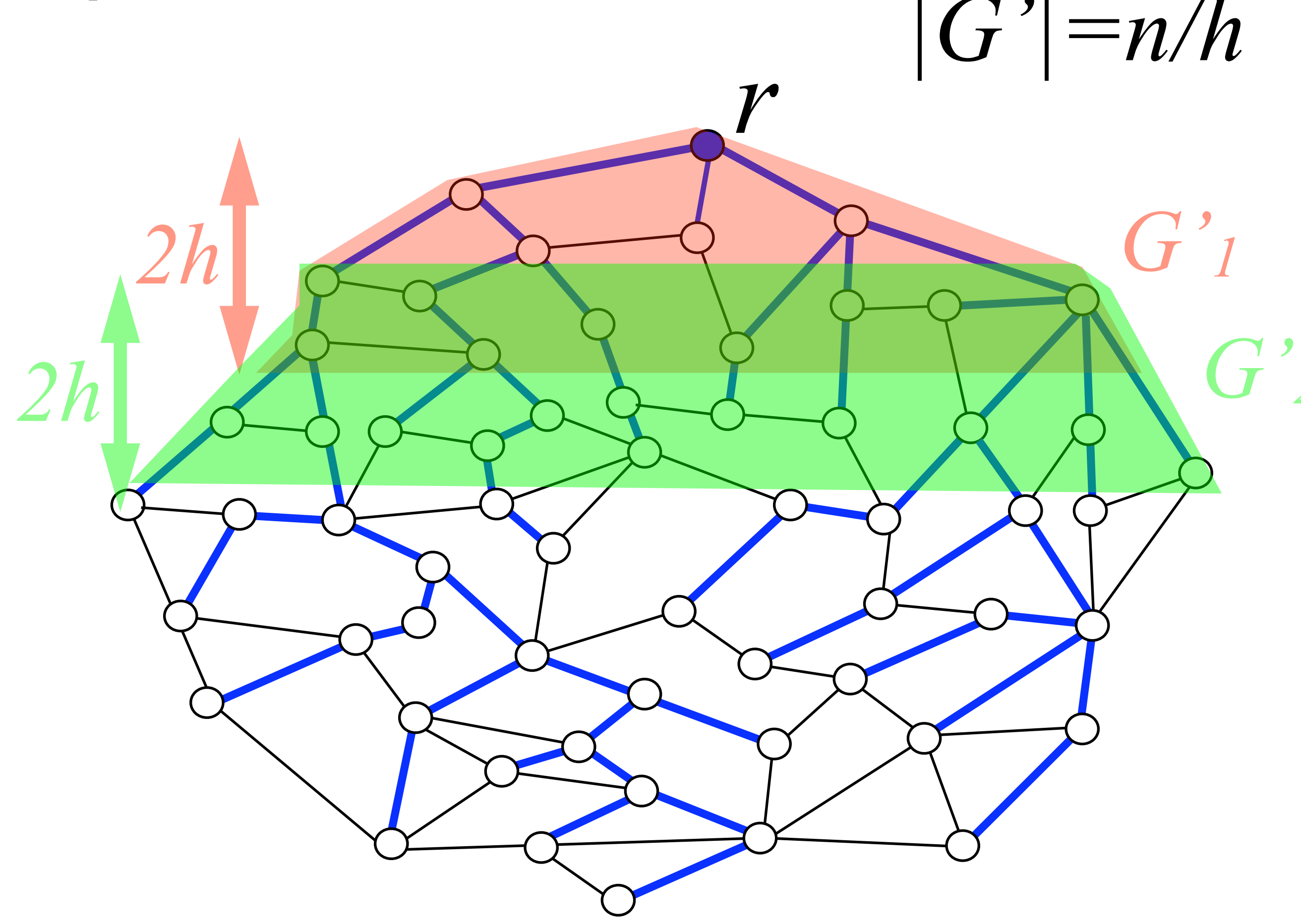

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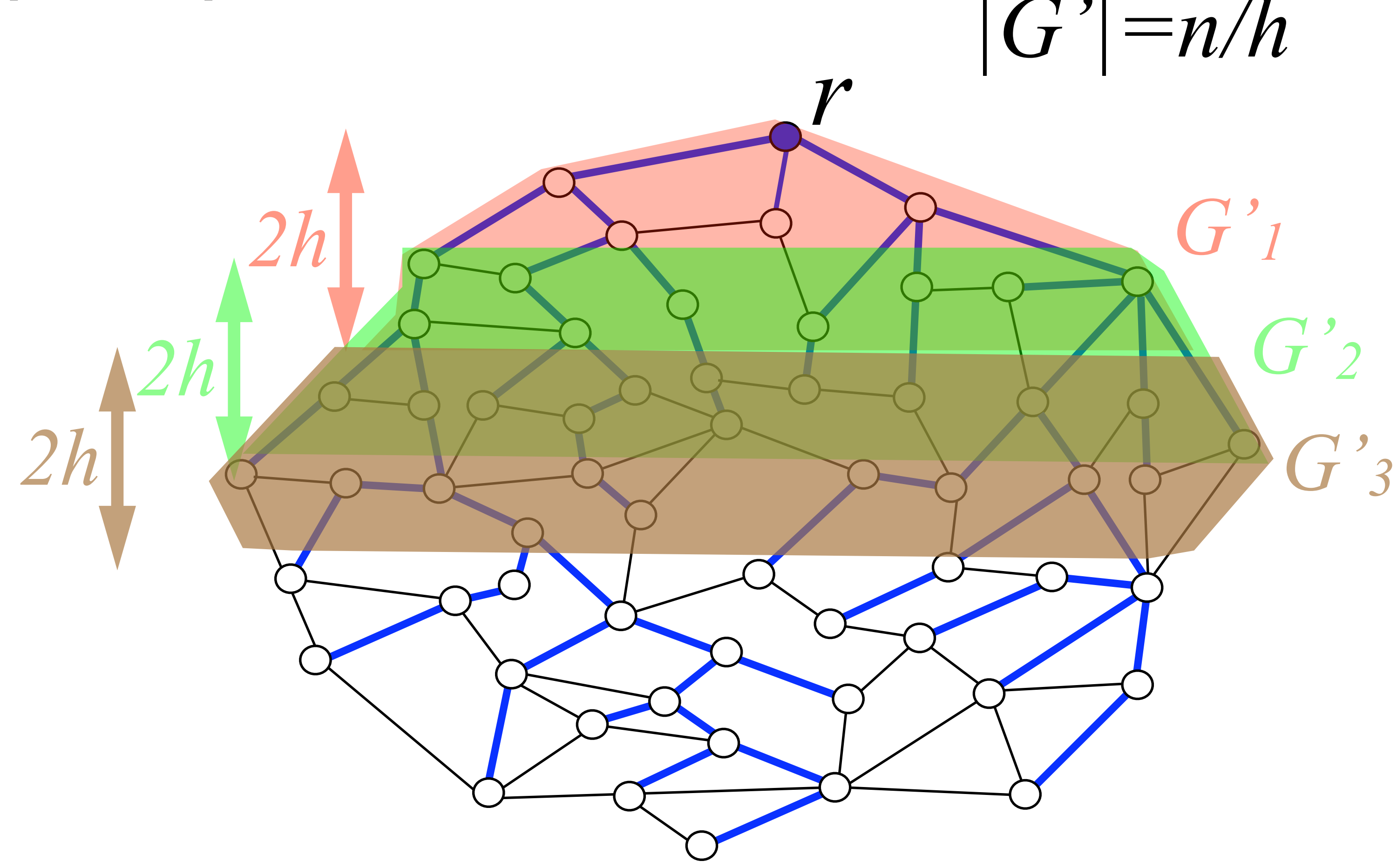

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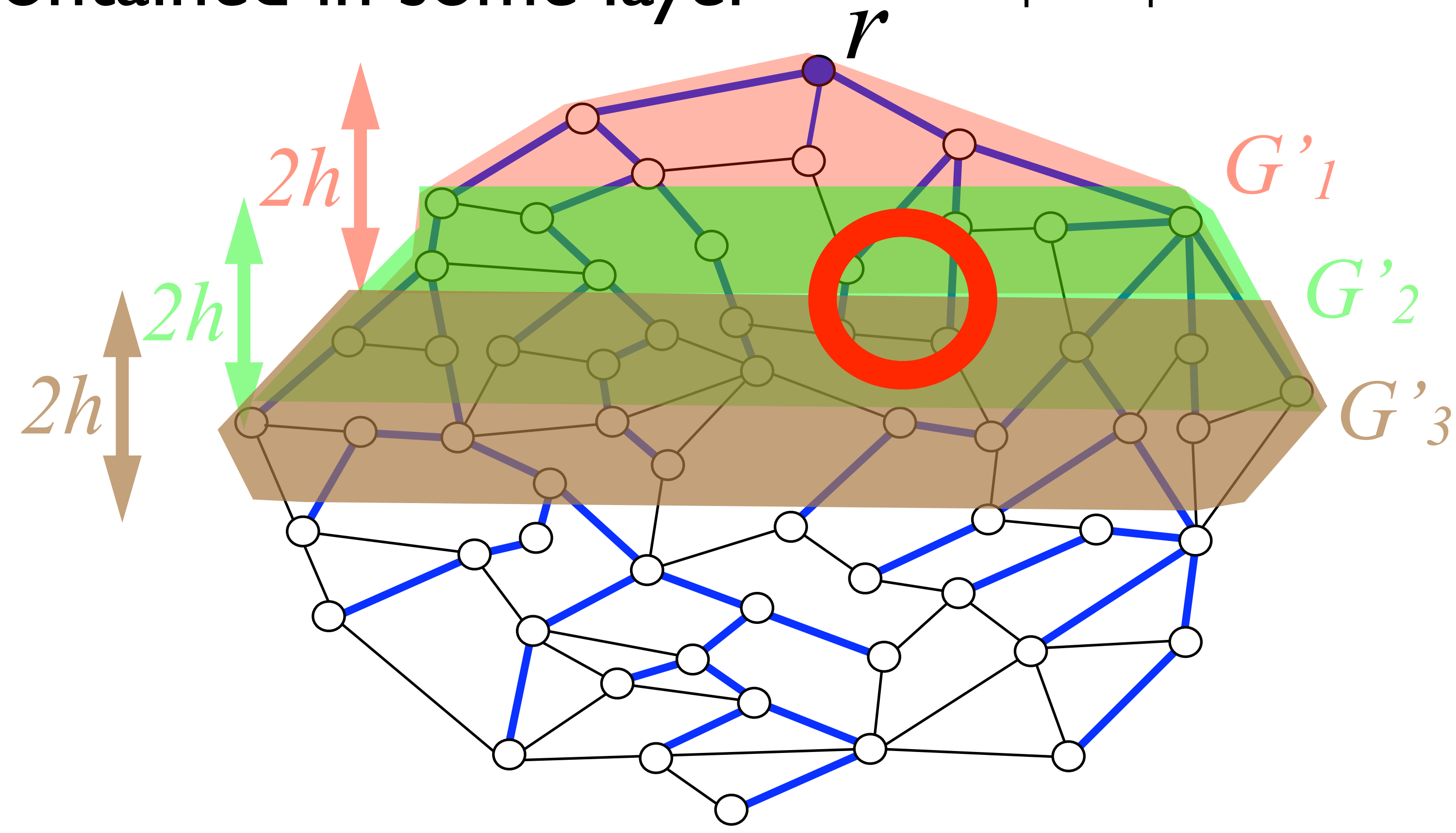

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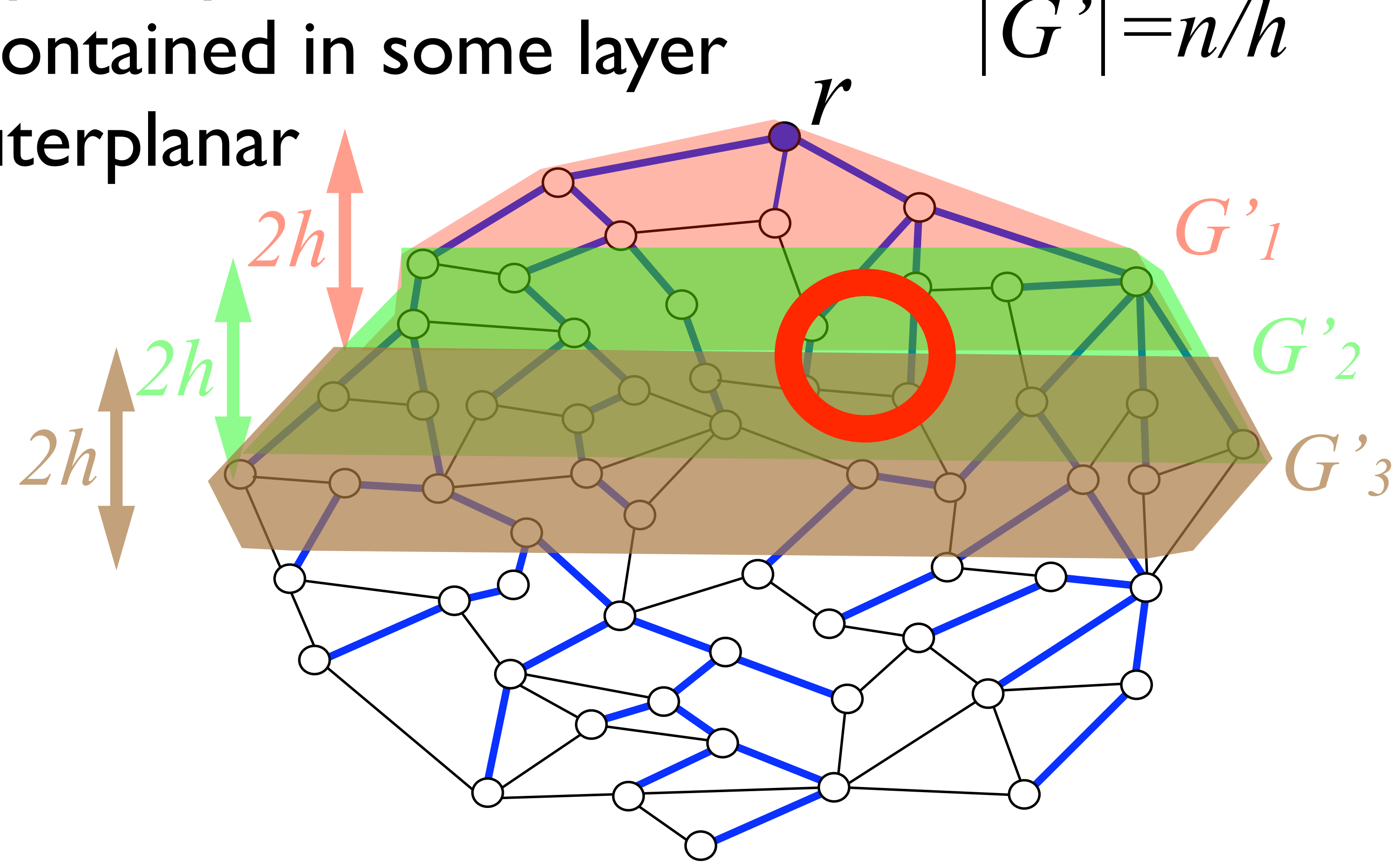

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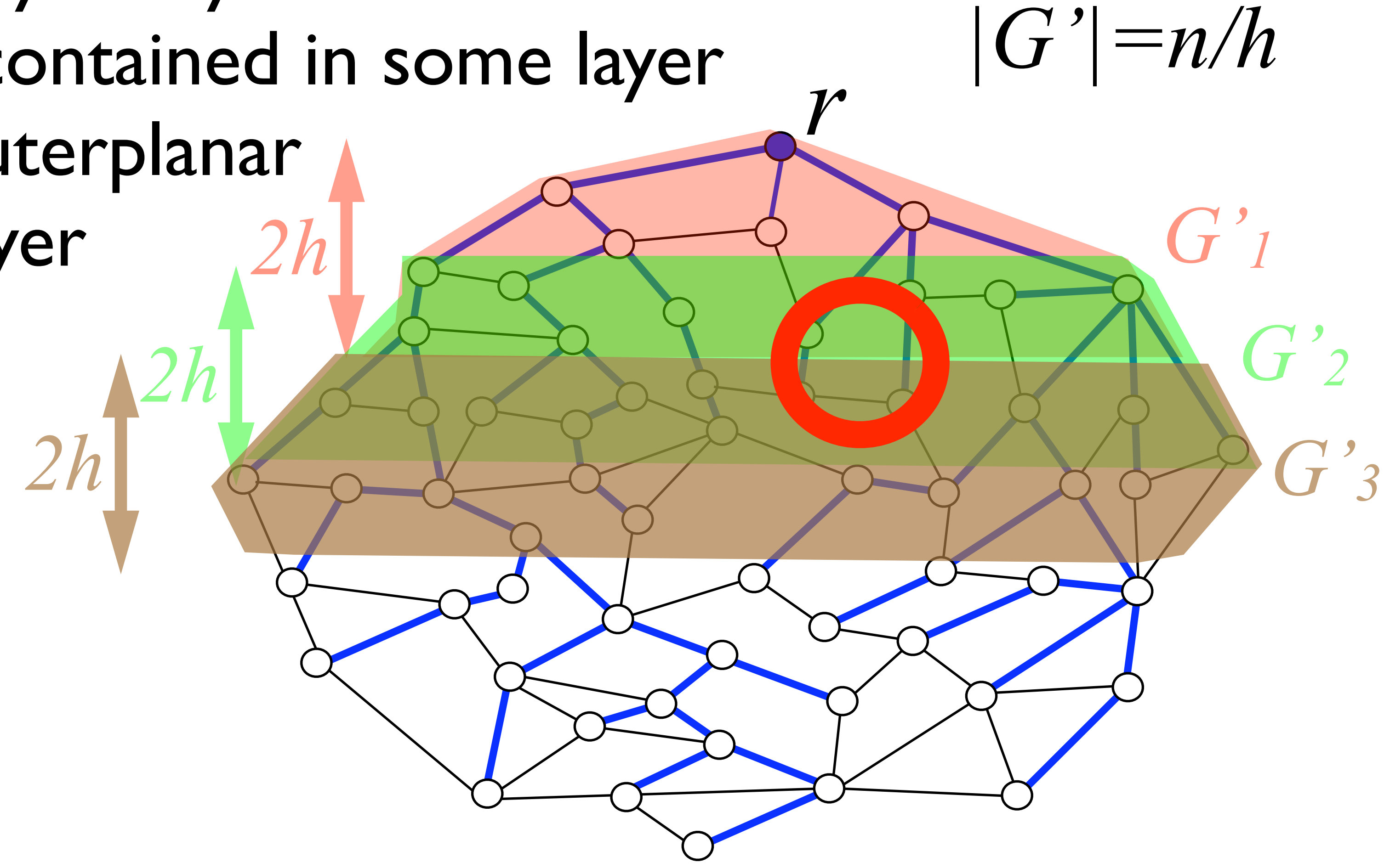

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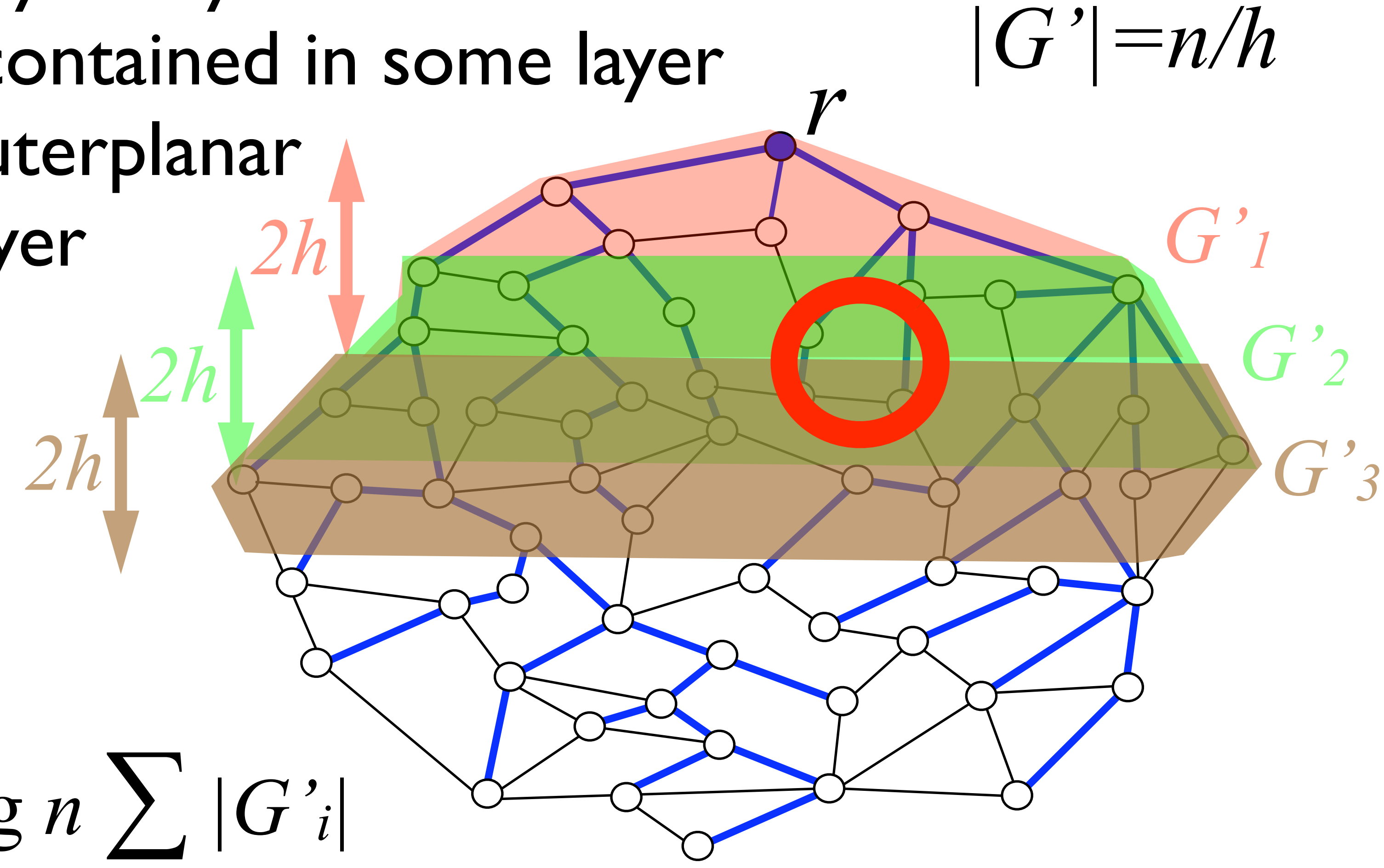

- *Time Complexity:*

$$\sum 2h\,|G'_i|\log|G'_i| \le 2h\log n \sum |G'_i|$$

$$= 2h\log n\;\Theta(|G'|)$$

$$= 2h\log n\;\Theta(n/h) = O(n\log n)$$

Open Problems:

- Weighted girth in $o(n \log^2 n)$
- Shortest cycle through every vertex in $o(n \log^2 n)$
- Girth of directed planar graph in $o(n^{3/2})$

Thank You!