

Binary Searching a Tree

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MIT, CSAIL

Joint work with

Shay Mozes (Brown University)

Krzysztof Onak (MIT)

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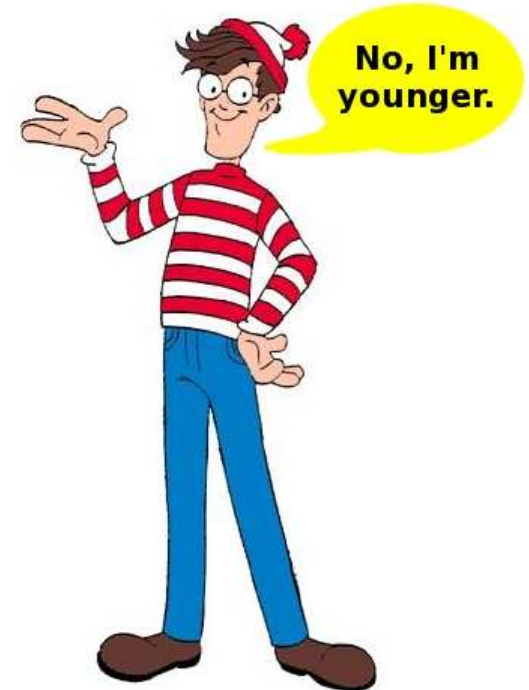
17 18 19 20 21 22 23 24



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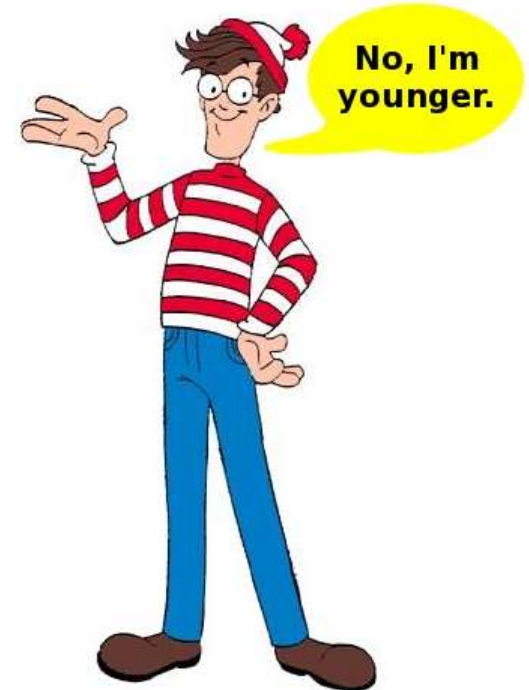
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You must be 19!

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- But there is a greater challenge to face!

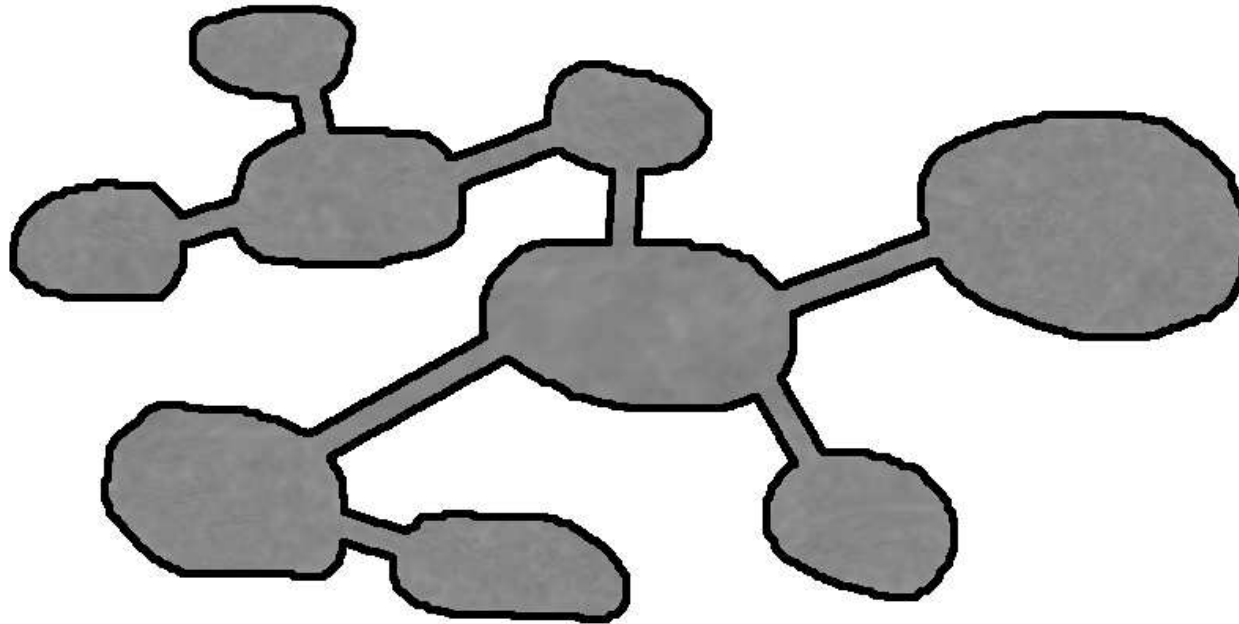
Searching in caves

- Waldo hides in a cave.



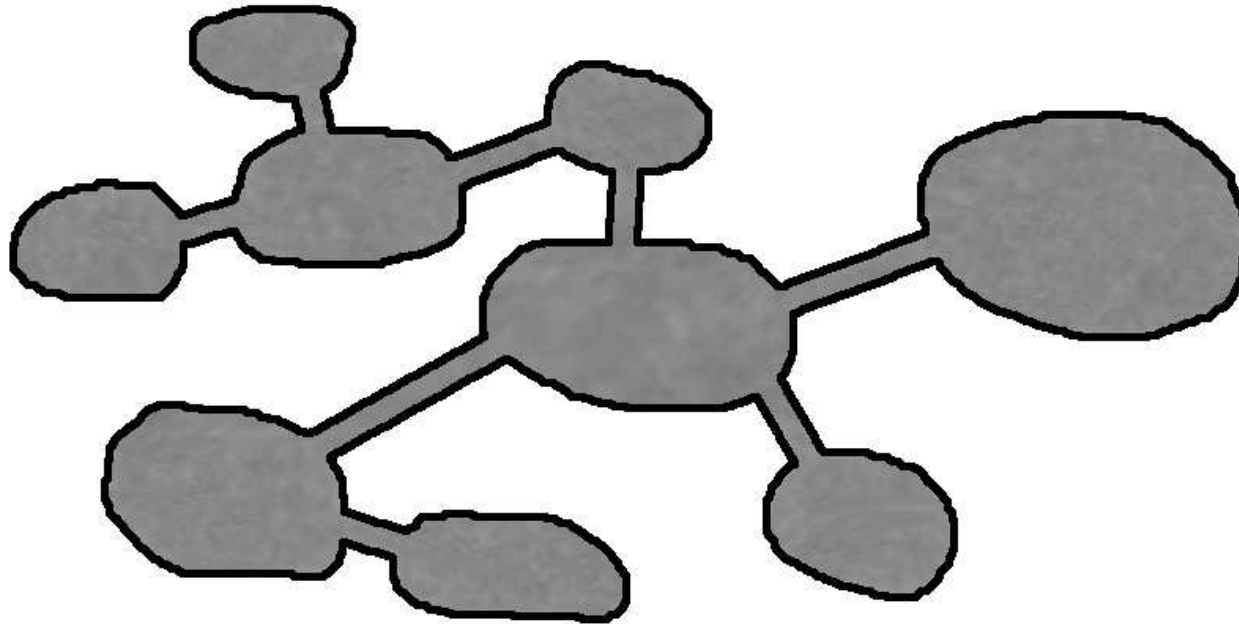
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- The cave consists of chambers and corridors.



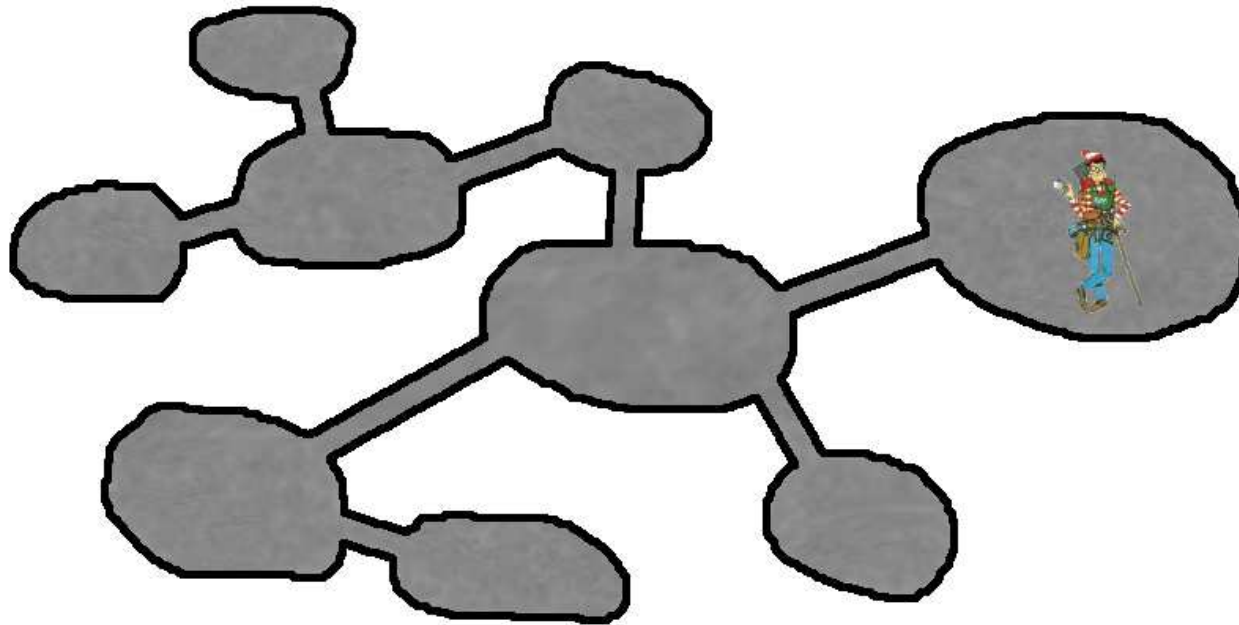
Searching in ~~caves~~ trees

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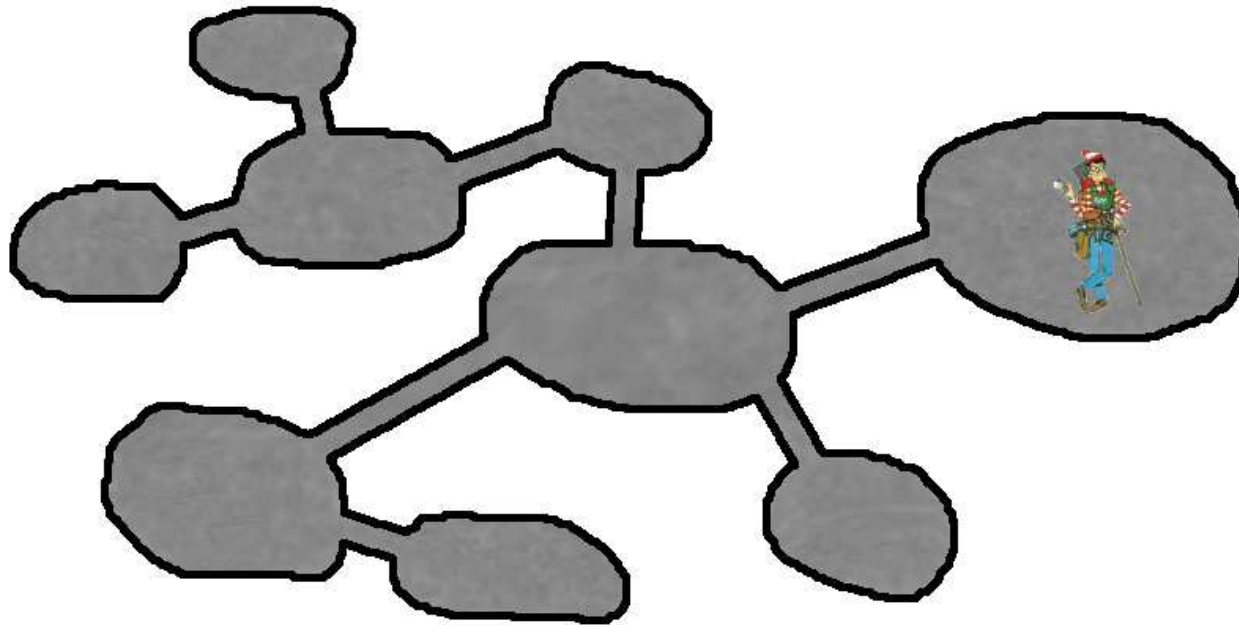
Searching in ~~caves~~ trees

- Waldo hides in a cave.
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- The graph of the cave is a tree.
- Goal: Figure out which chamber Waldo is in.



Two query models

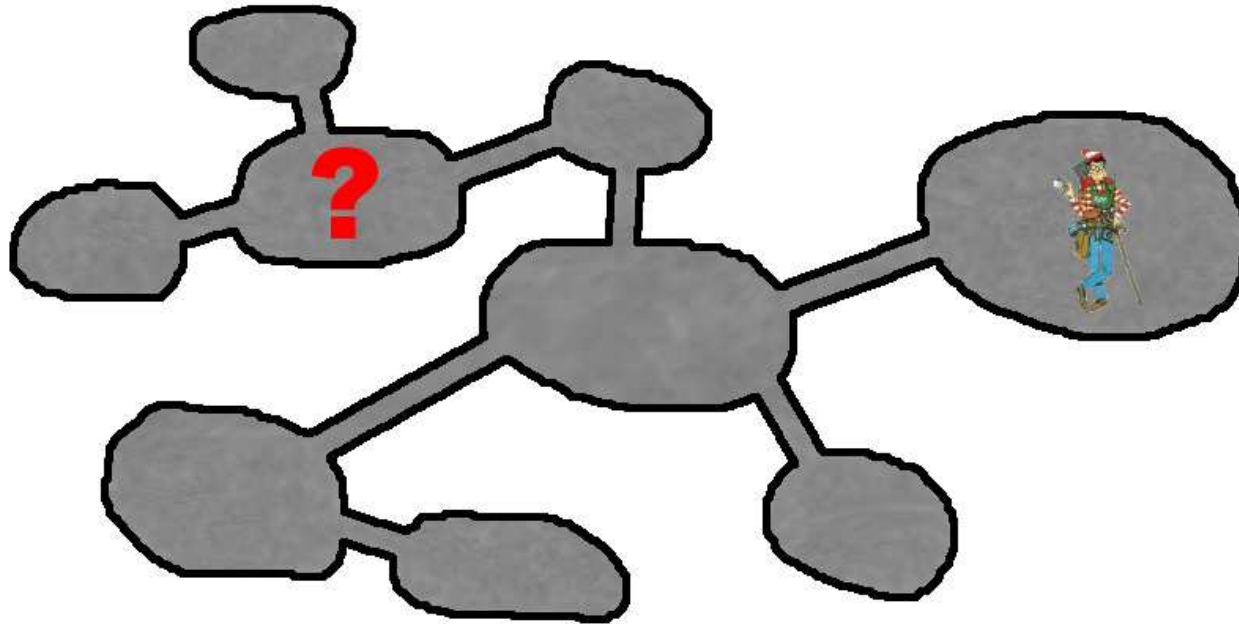
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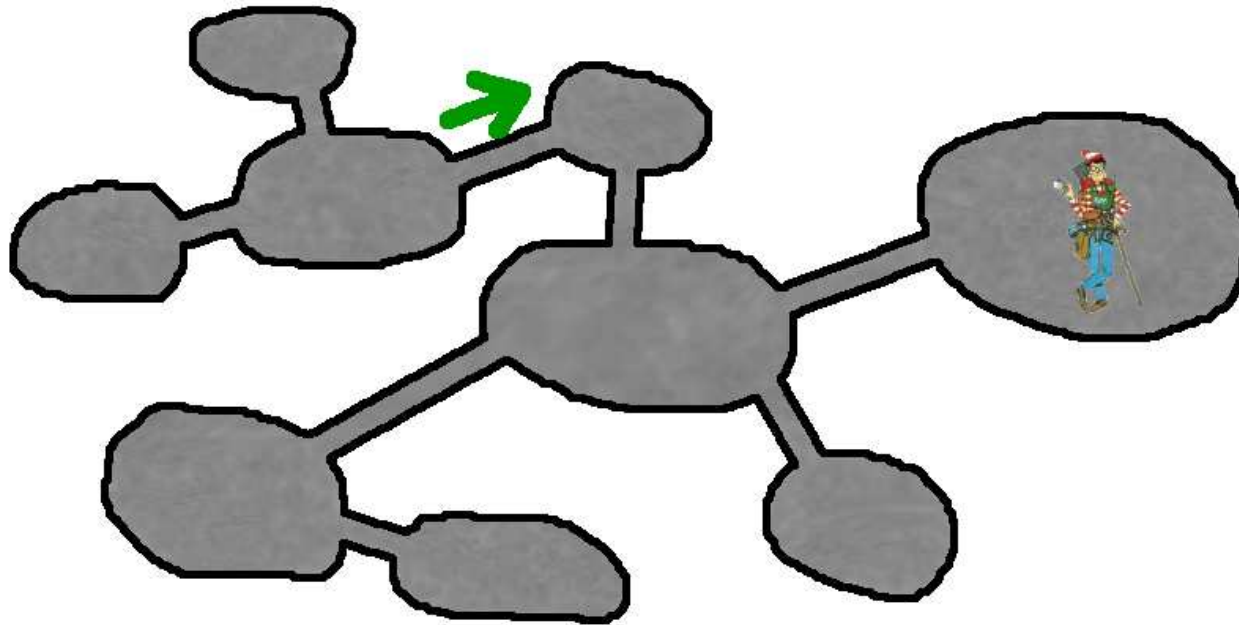
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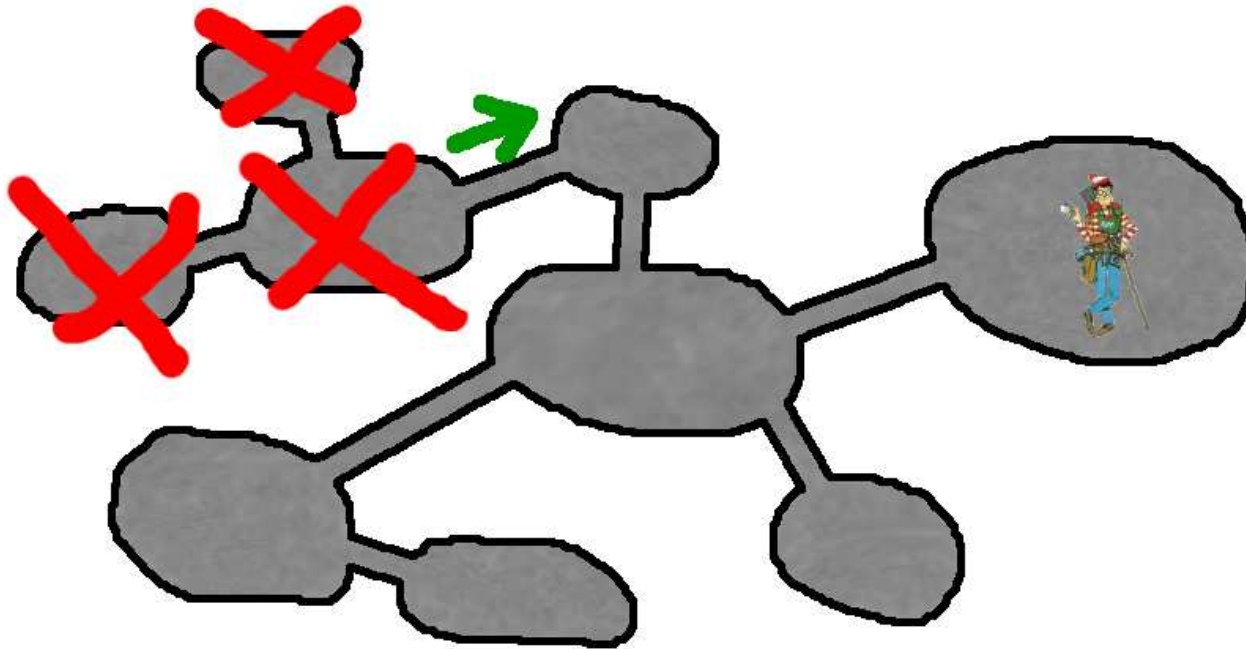
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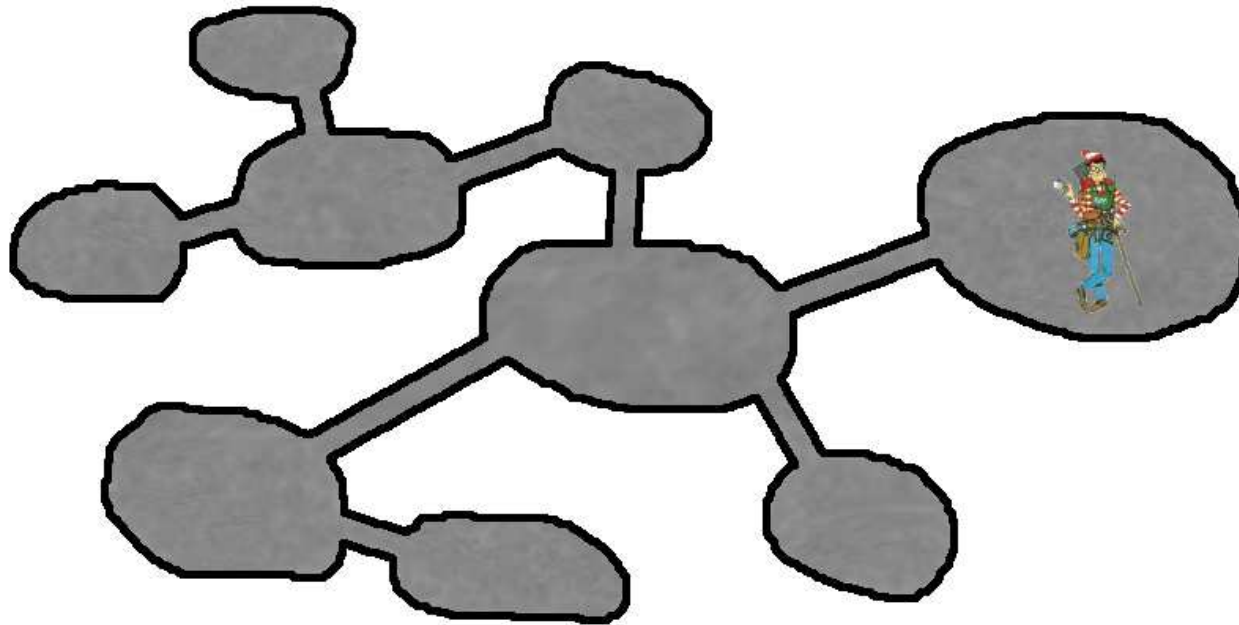
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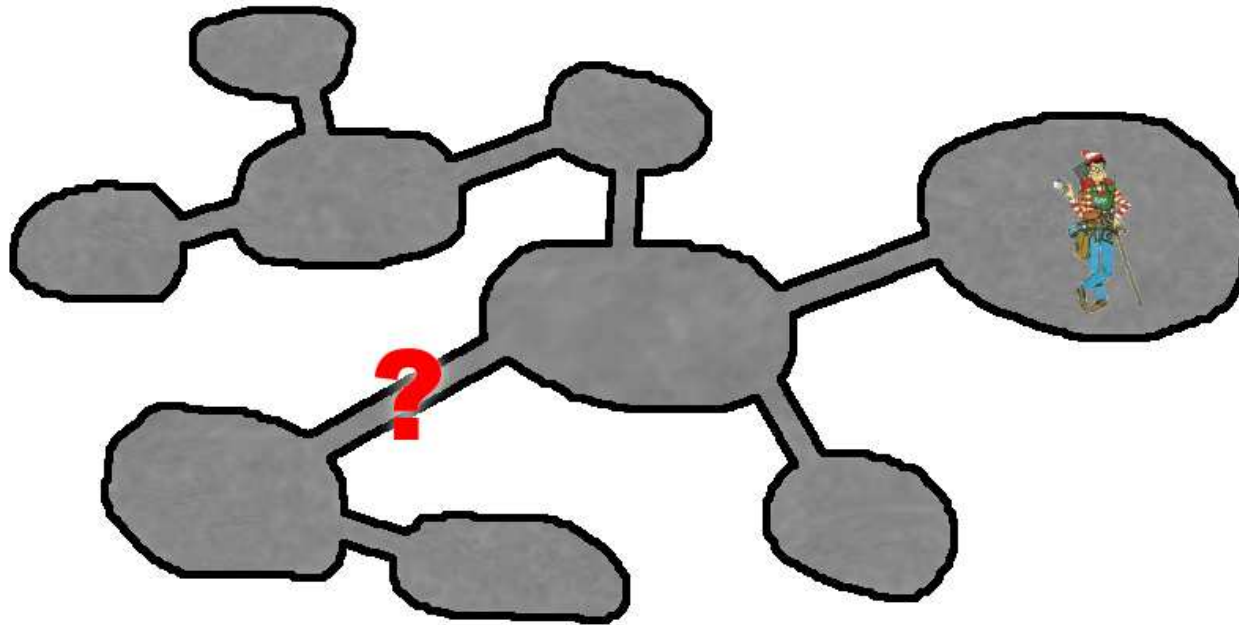
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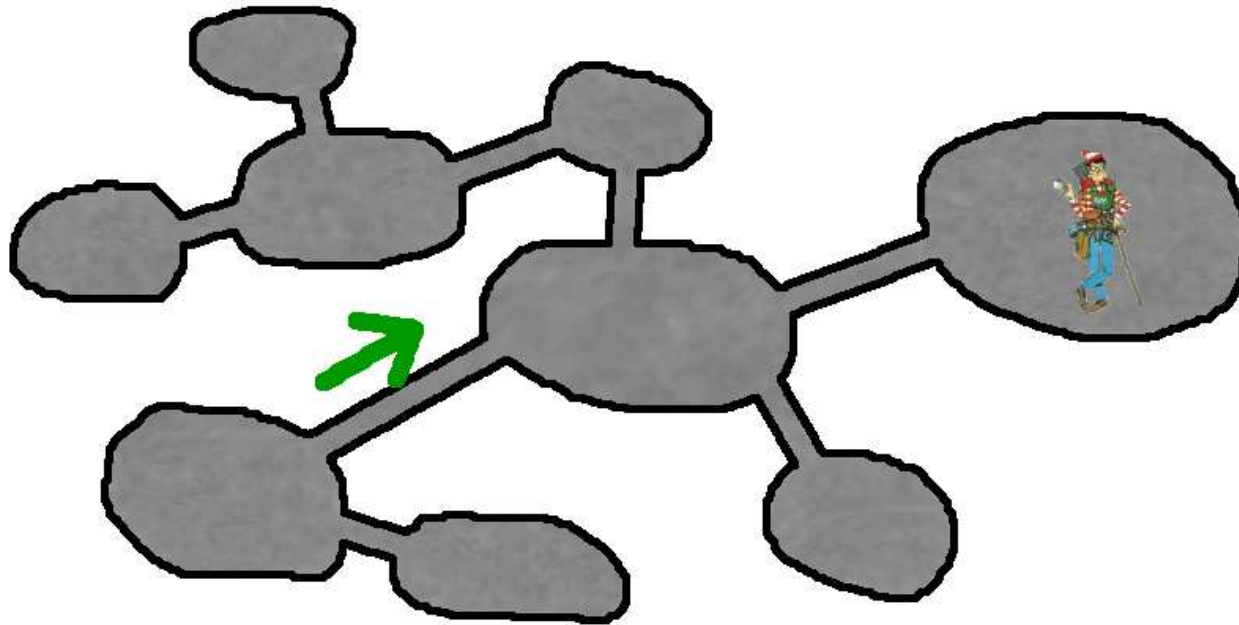
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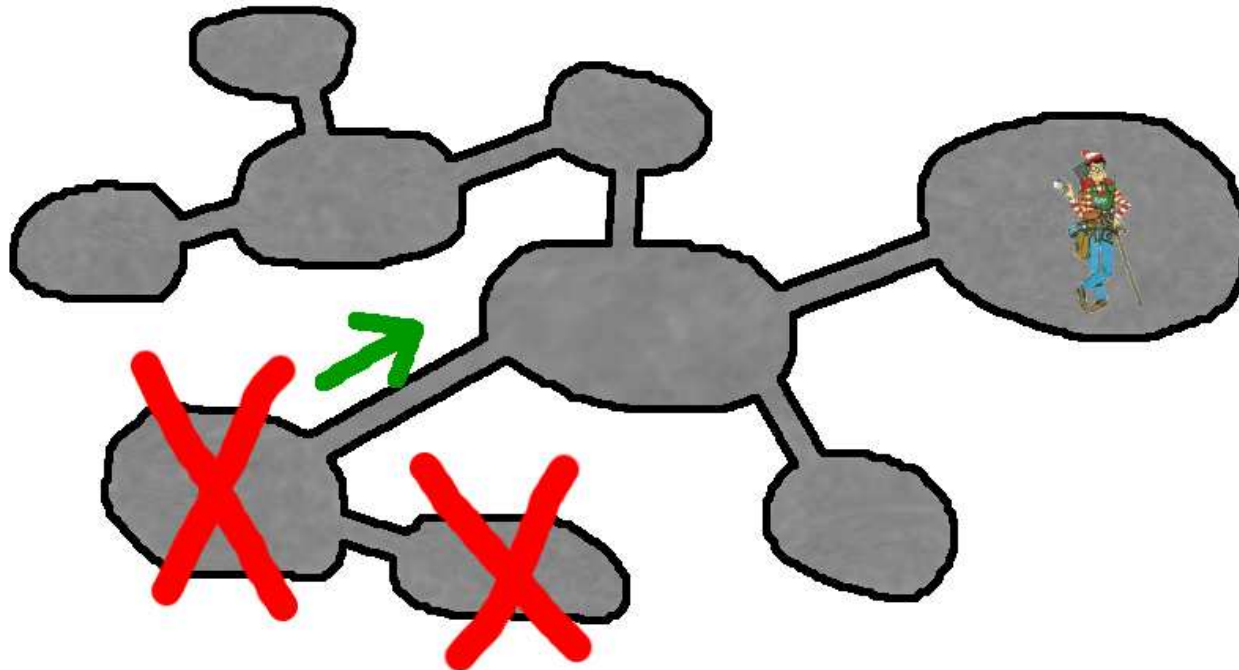
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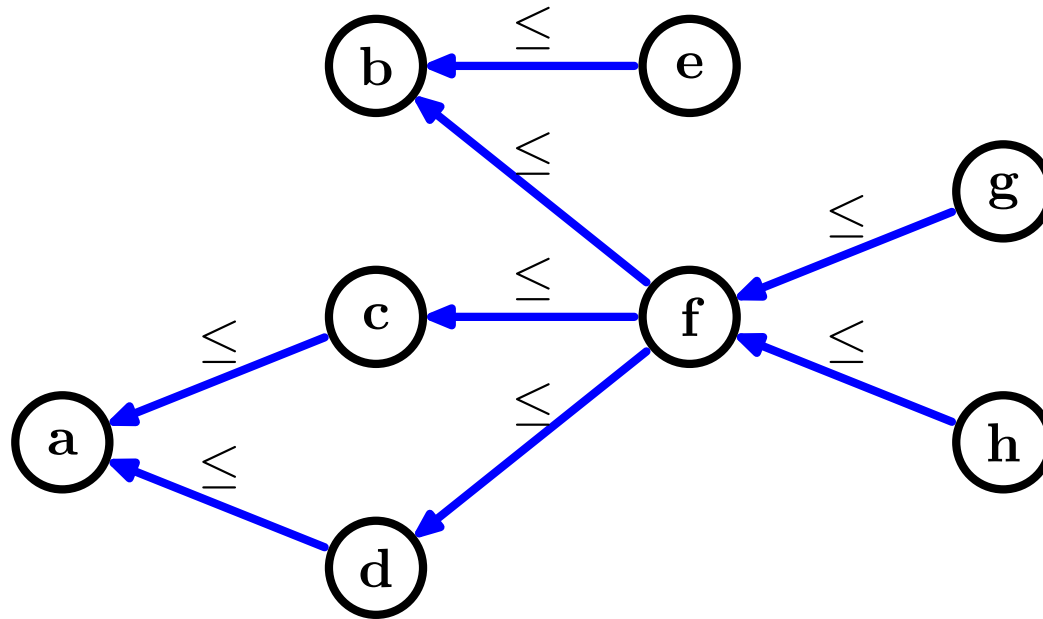
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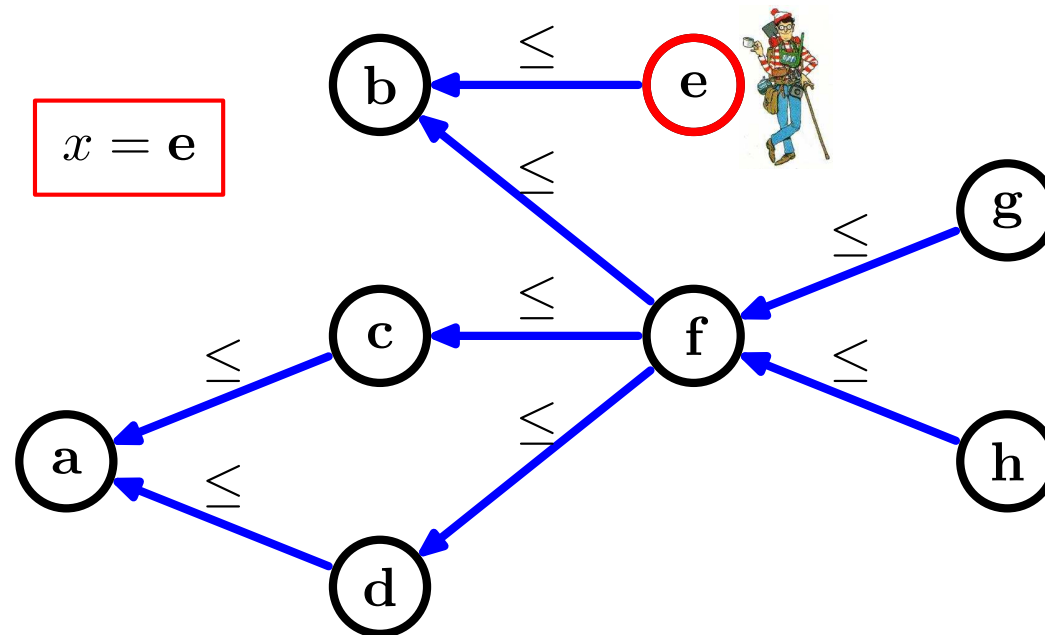
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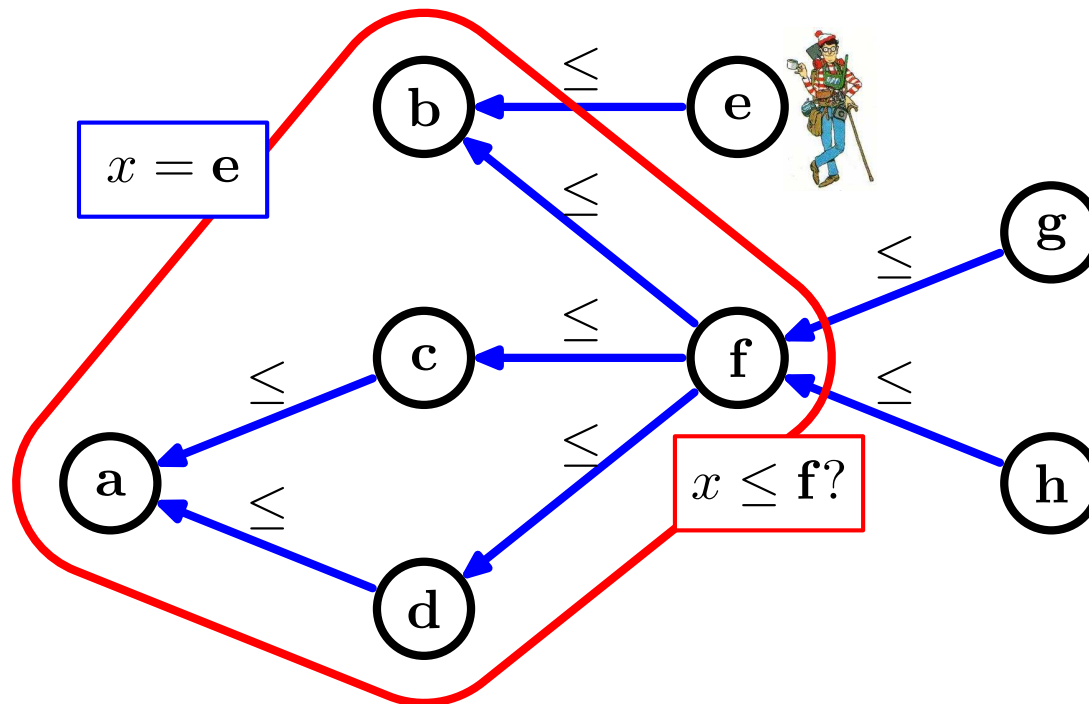
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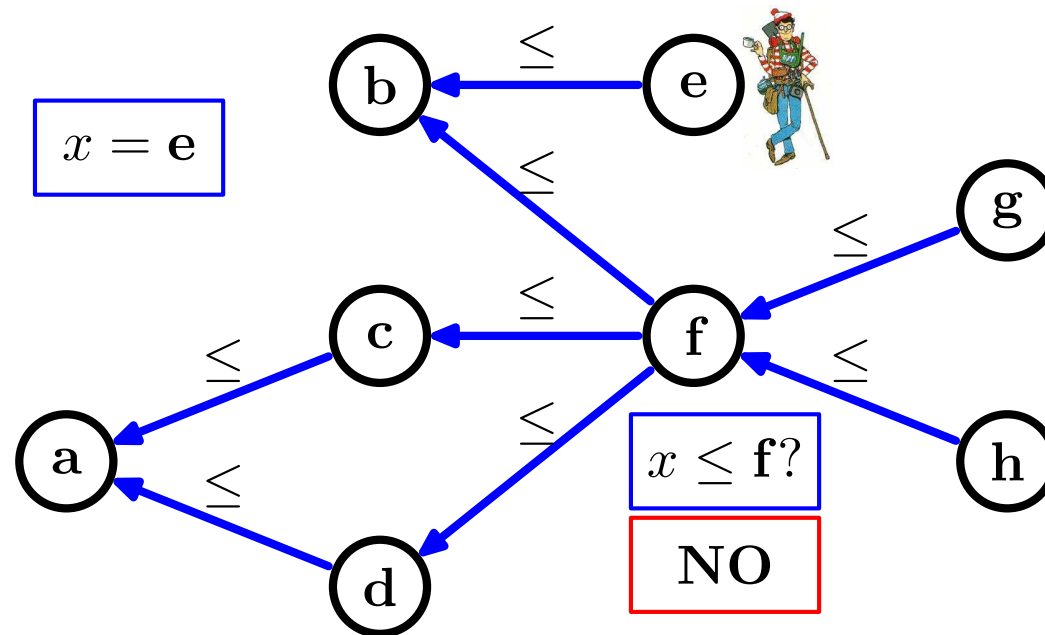
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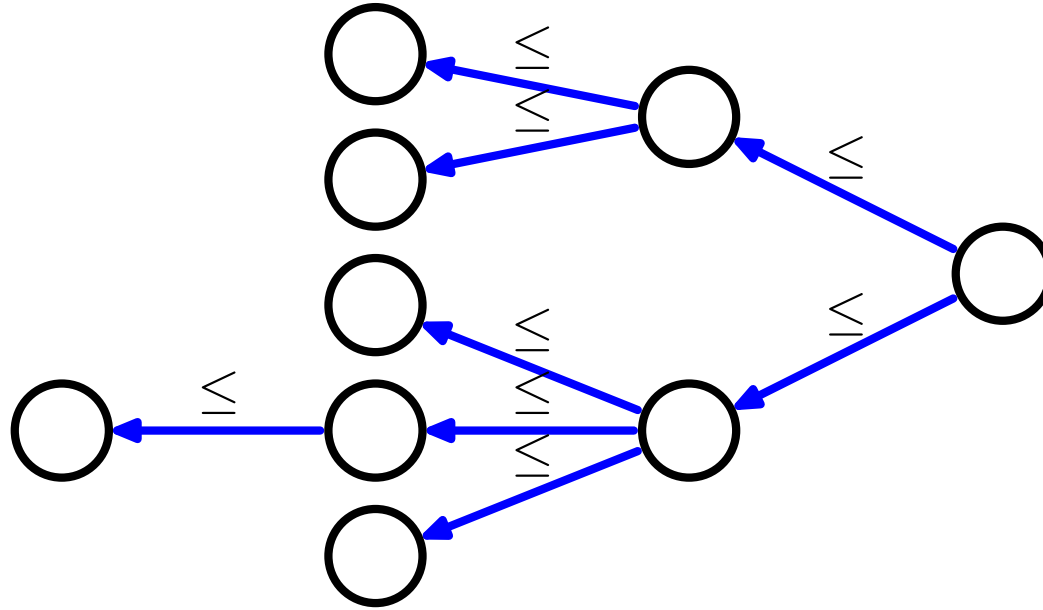
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- For some posets the problem is identical to searching in trees in the edge-query model.



Optimal strategies

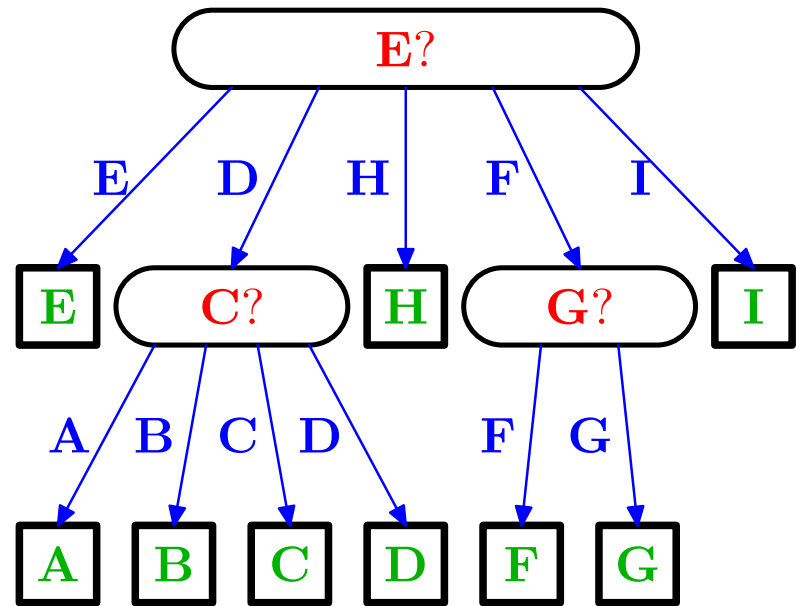
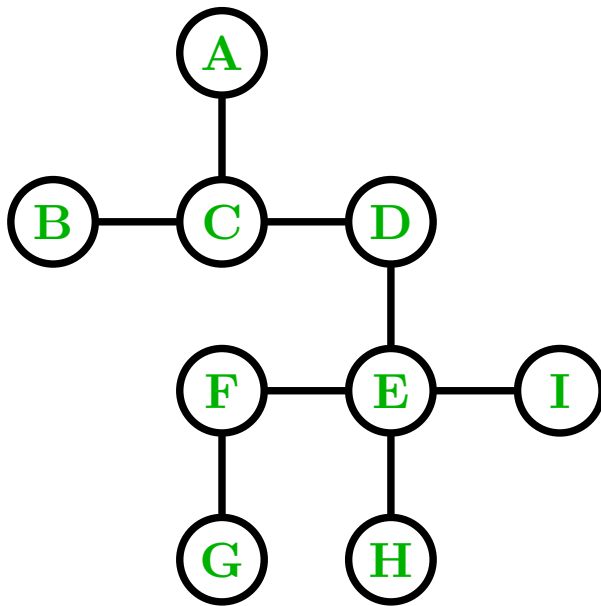
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- A sample optimal strategy in the vertex-query model:



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Our Results

- $O(n)$ in the edge-query model [SODA 2008]
 - novel bottom-up construction algorithm
 - a method for reusing parts of already computed subproblems
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- Applications
 - file system synchronization
 - bug detection

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We start with the vertex-query model.

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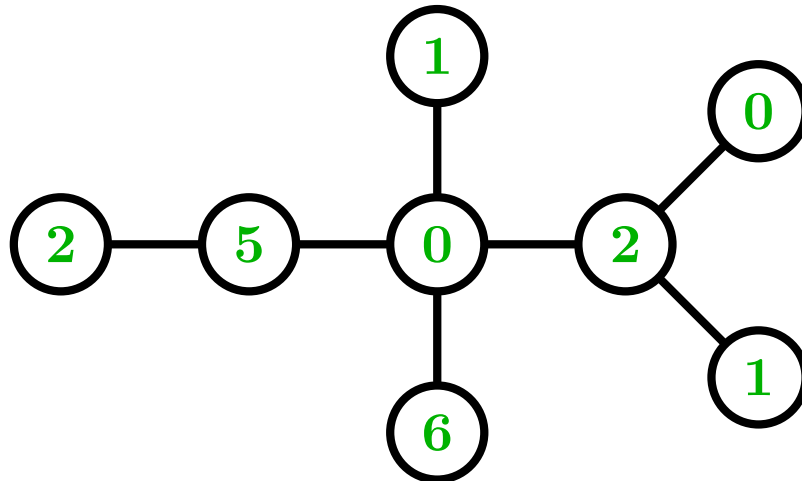
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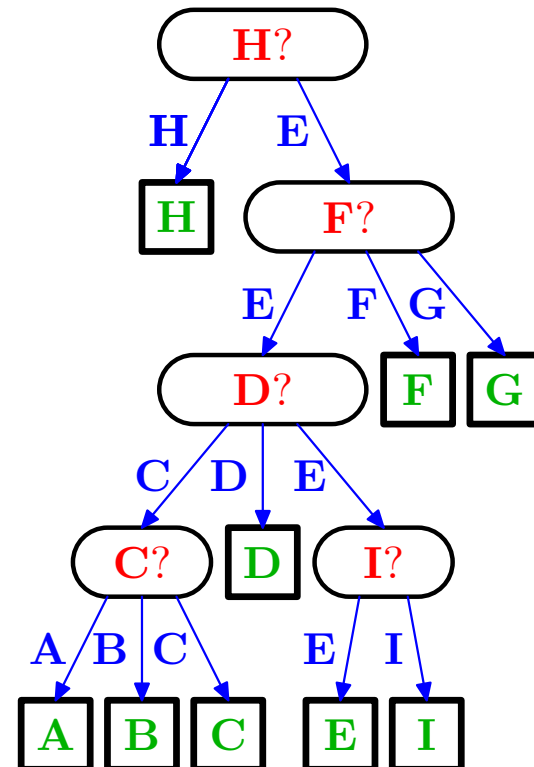
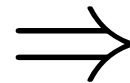
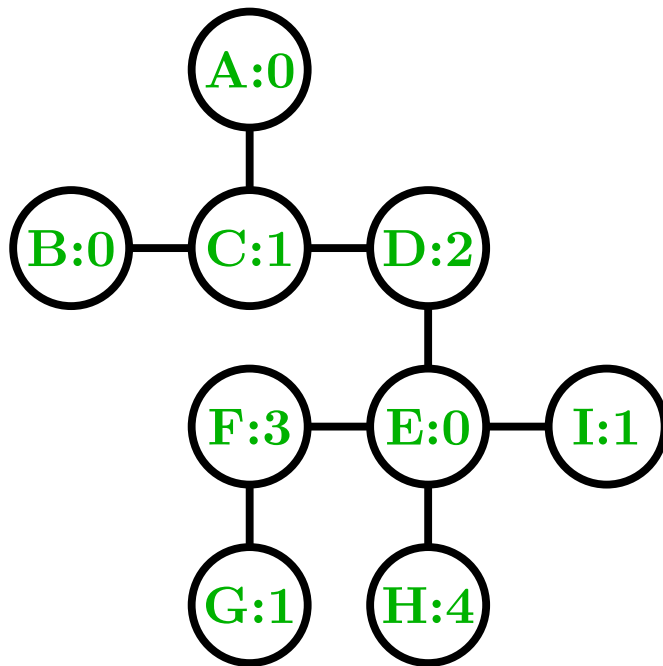
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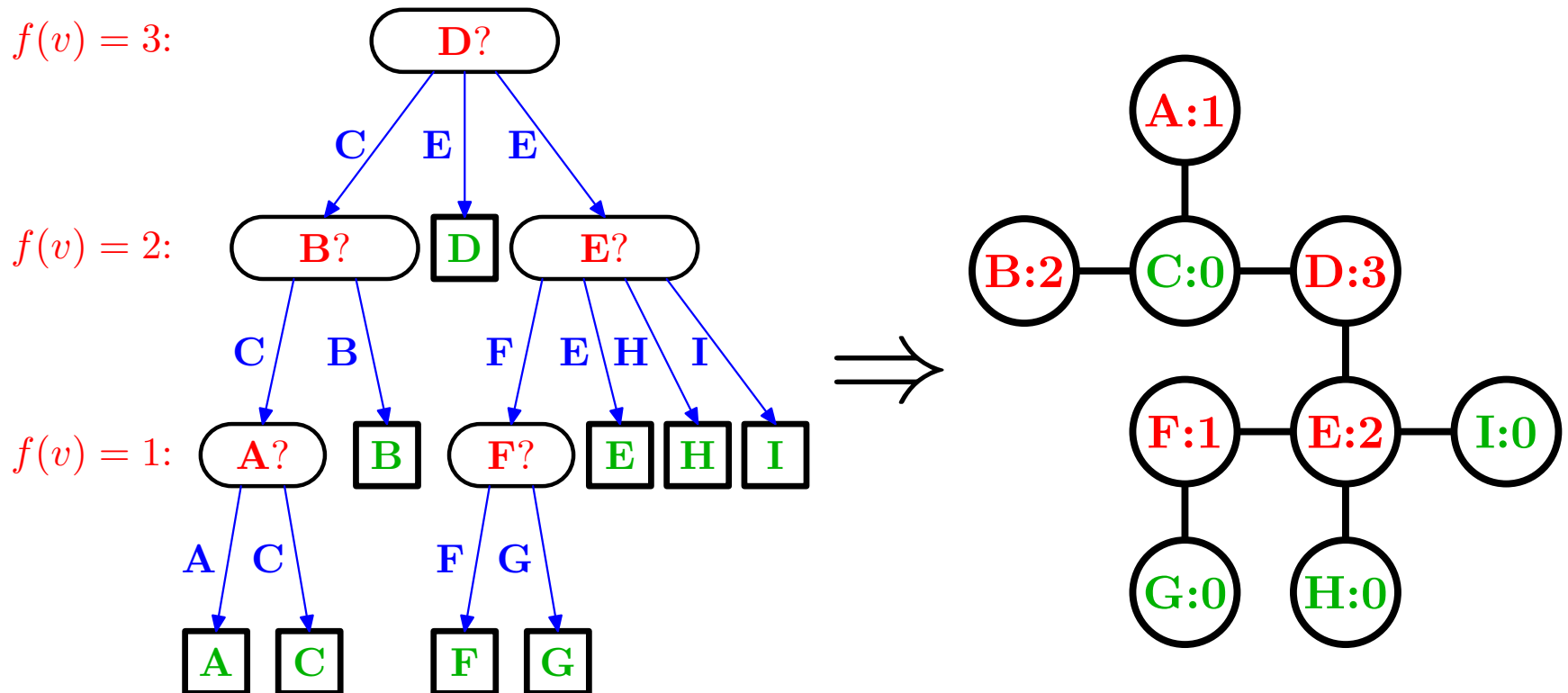
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Conclusion

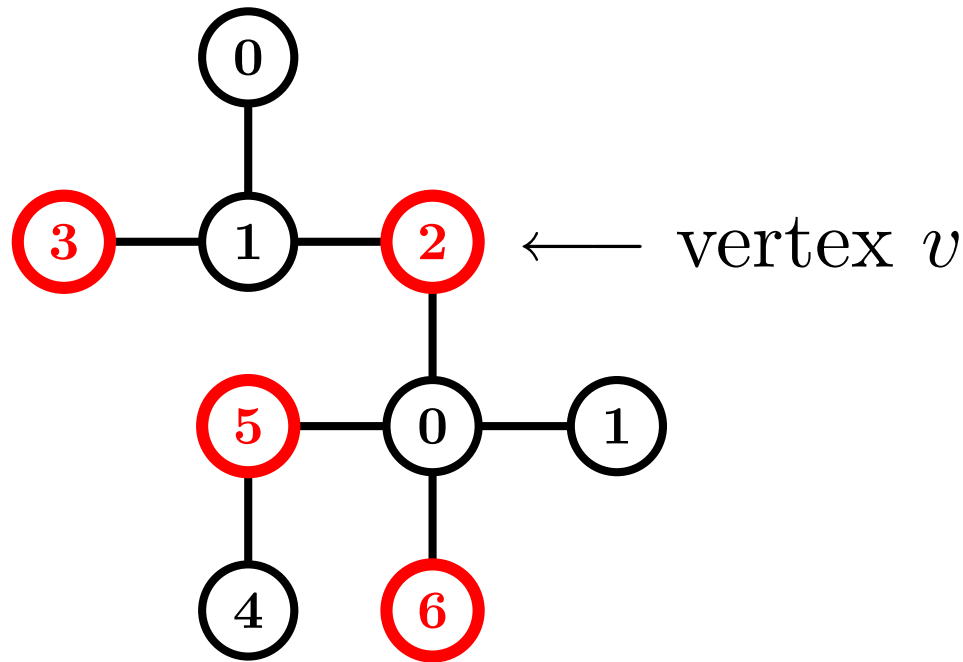
It suffices to construct a strategy
function of the least maximum!

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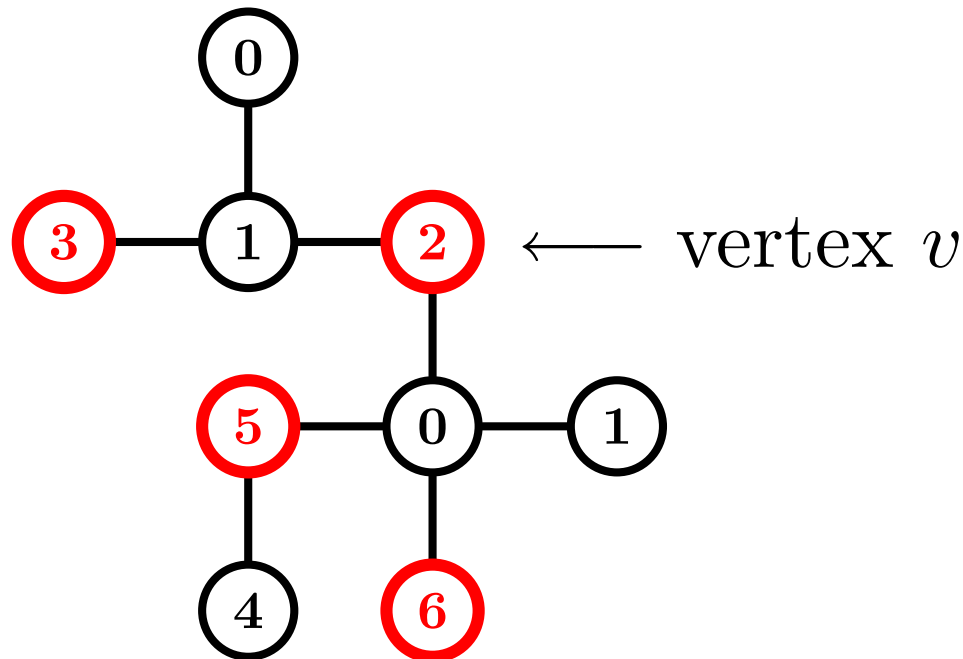
Values visible from v : 3, 2, 5, 6

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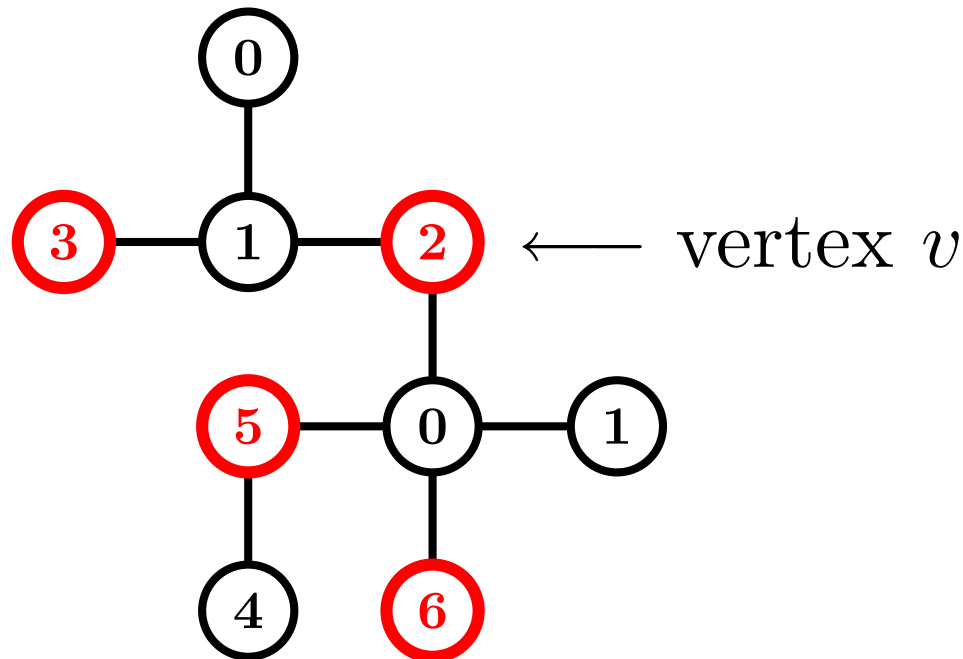
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The visibility sequences are ordered lexicographically.
For instance, $(8, 4, 3, 2) > (7, 6, 4, 2, 1)$.

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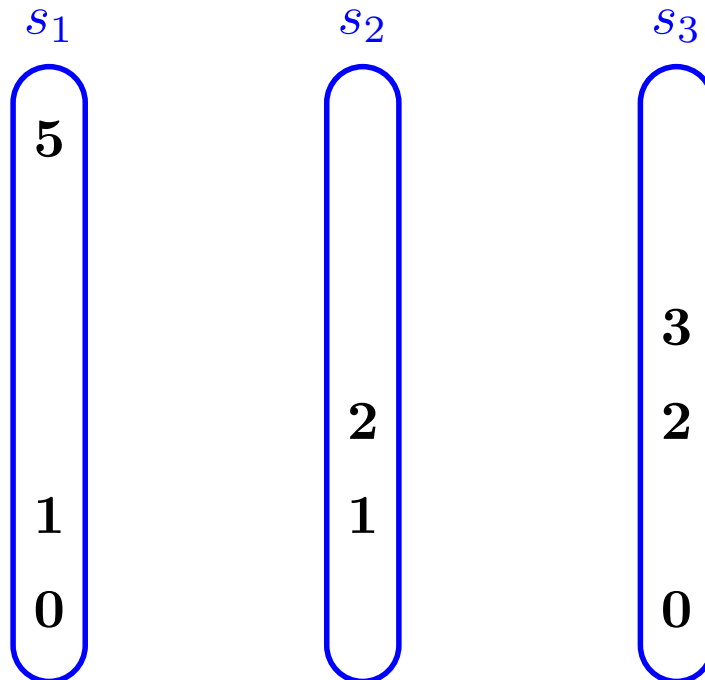
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- To get a correct strategy function, it suffices to know the visibility sequences from children of v in their subtrees.
 - An **extension operator** is a procedure that takes those visibility sequences, extends the function, and returns the visibility sequence from v in the subtree rooted at v .

An Optimal Extension

- A *minimizing* extension is one that gives the lexicographically smallest visibility sequence at v .
 - *minimizing extensions* accumulate to an optimal solution [OP 2006].

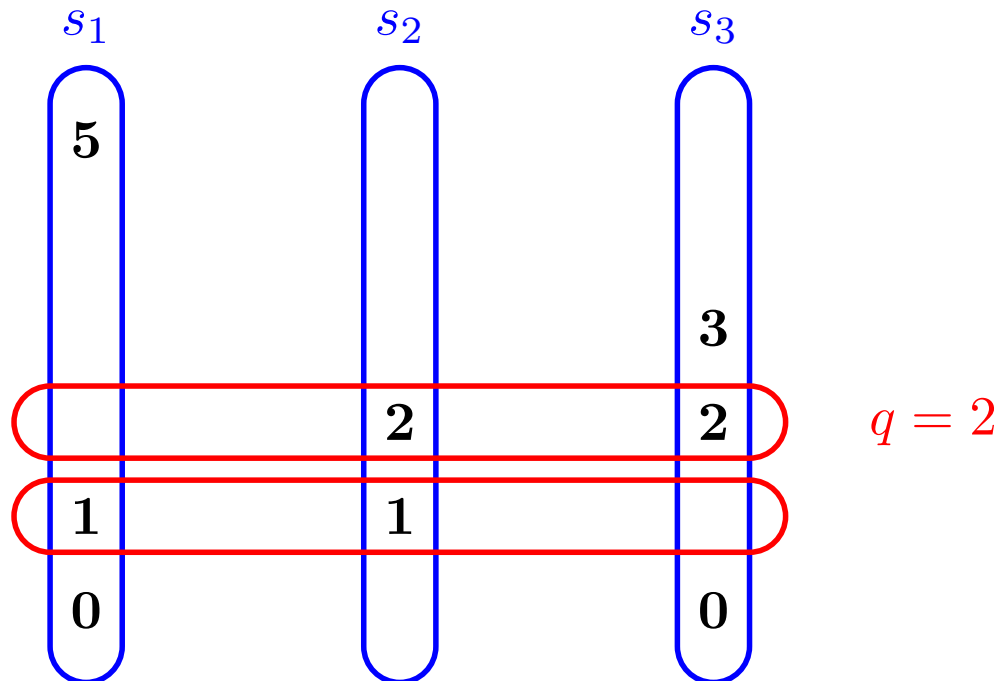
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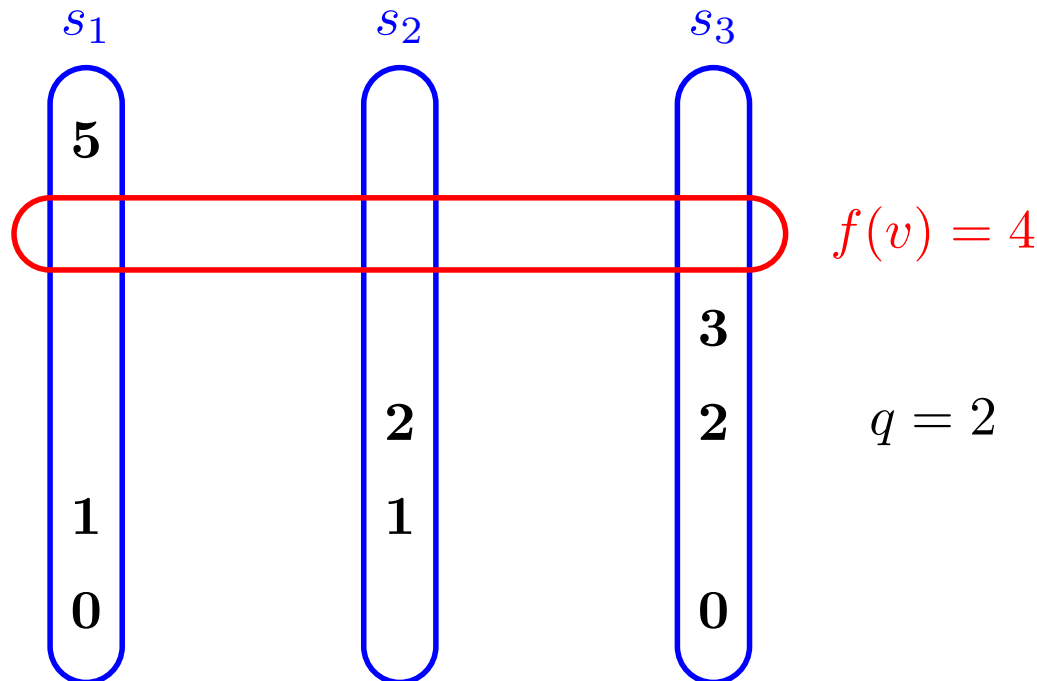
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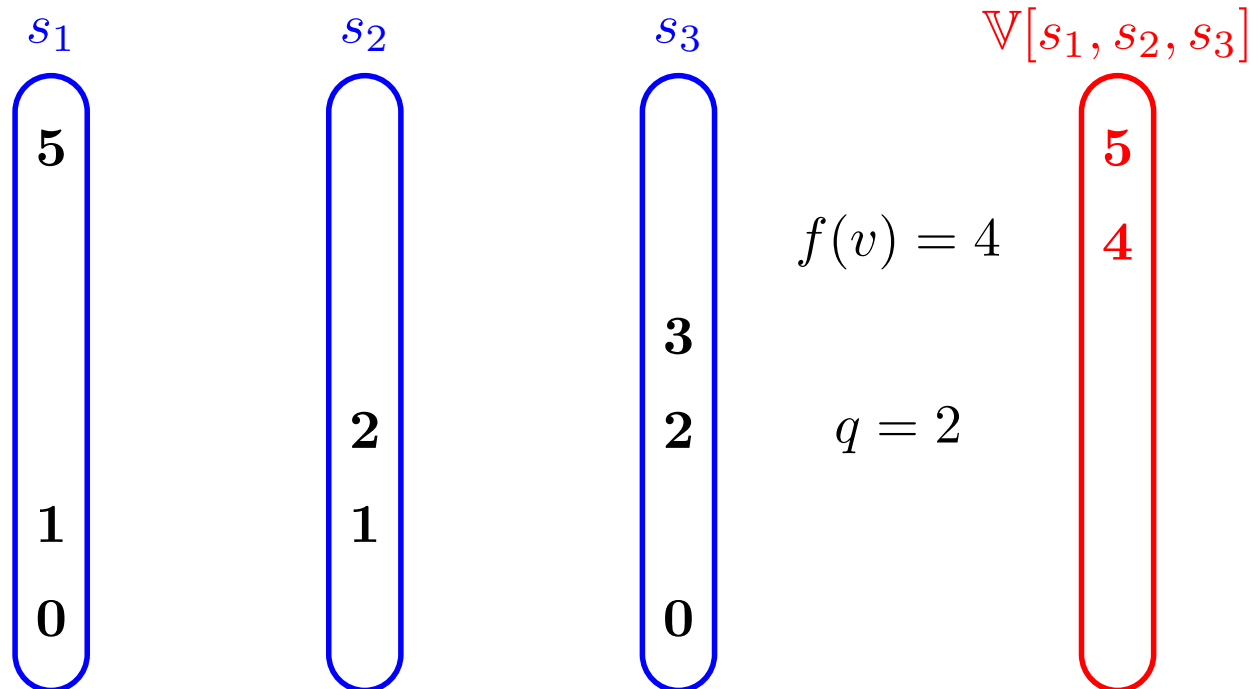
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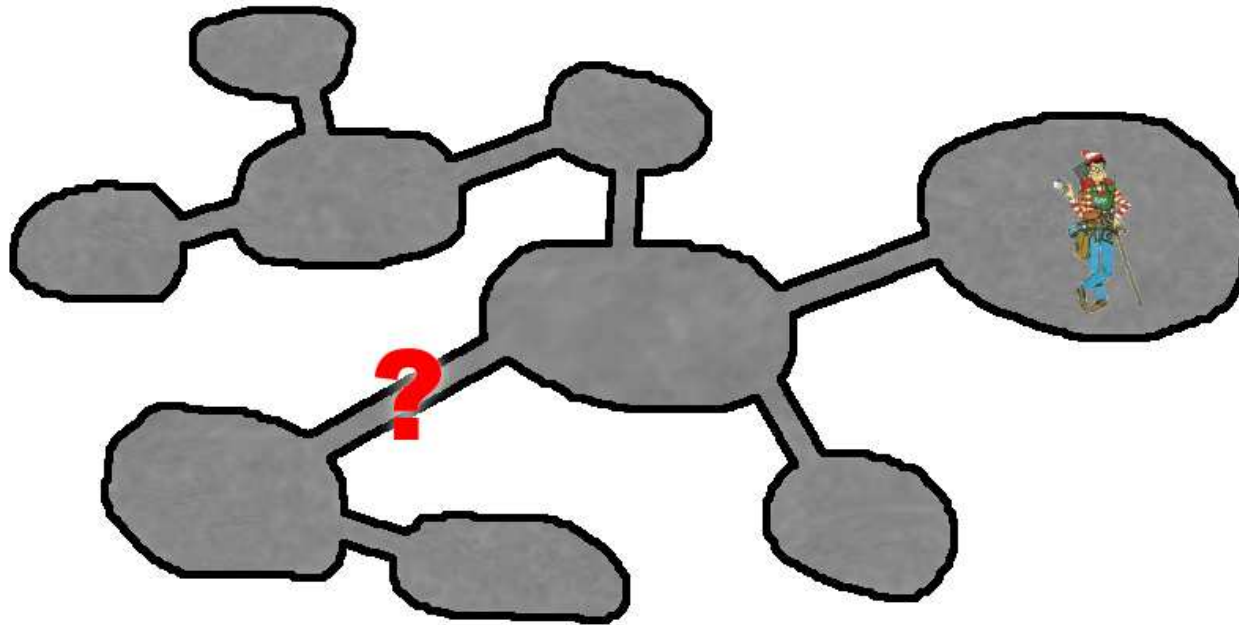
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- The running time can be improved to $O(n)$ fairly simple.

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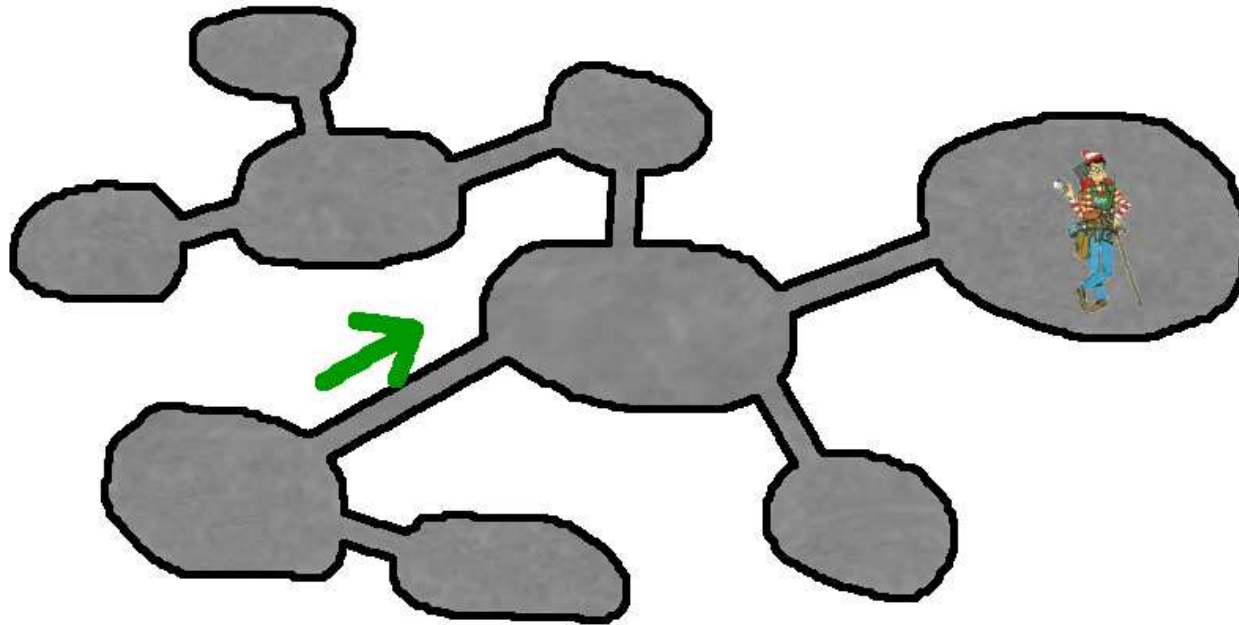
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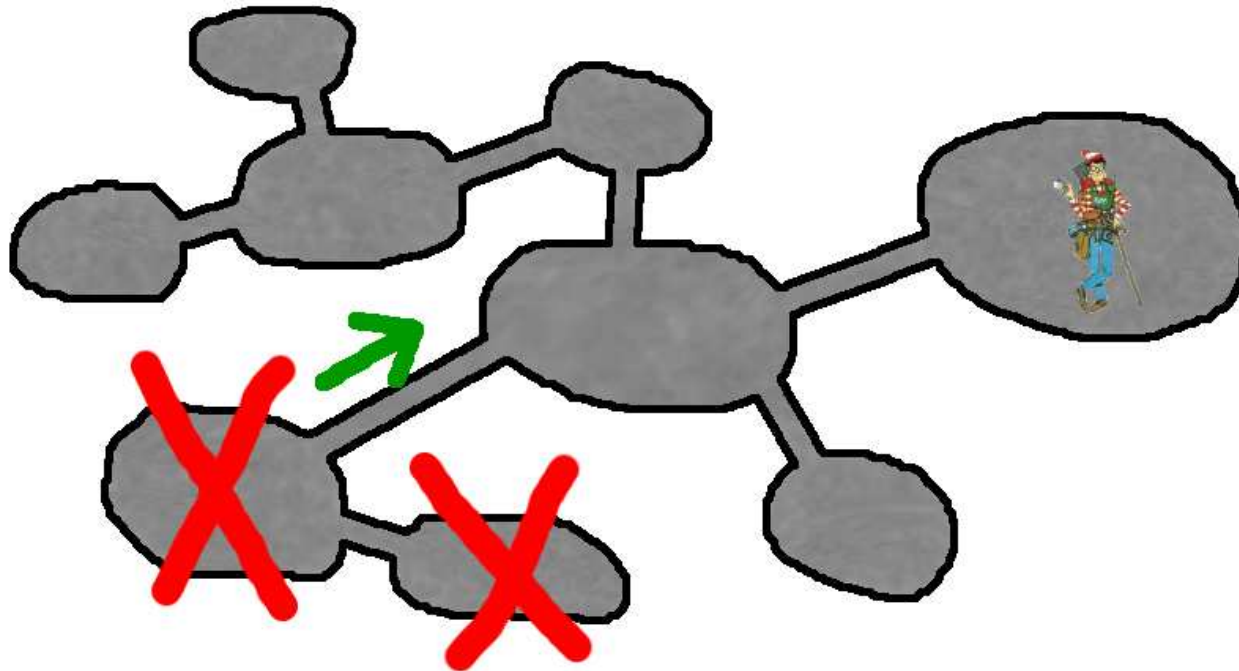
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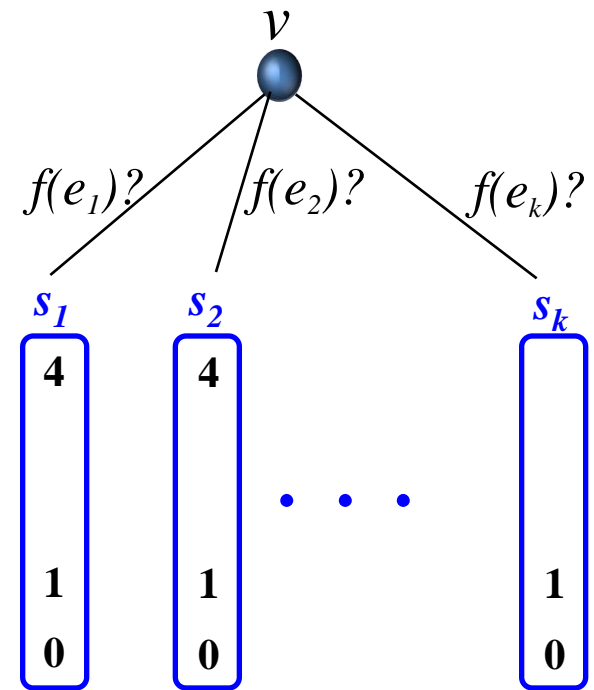
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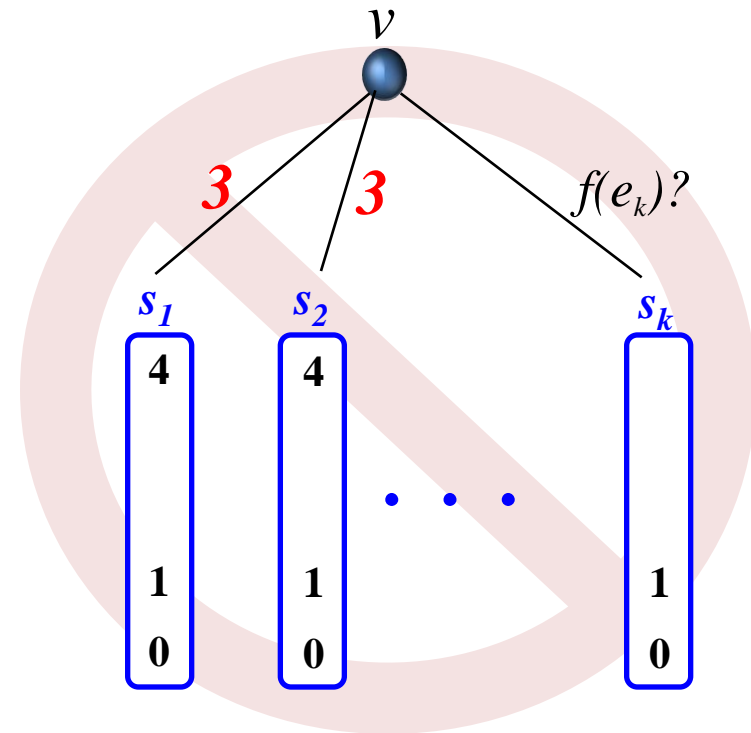
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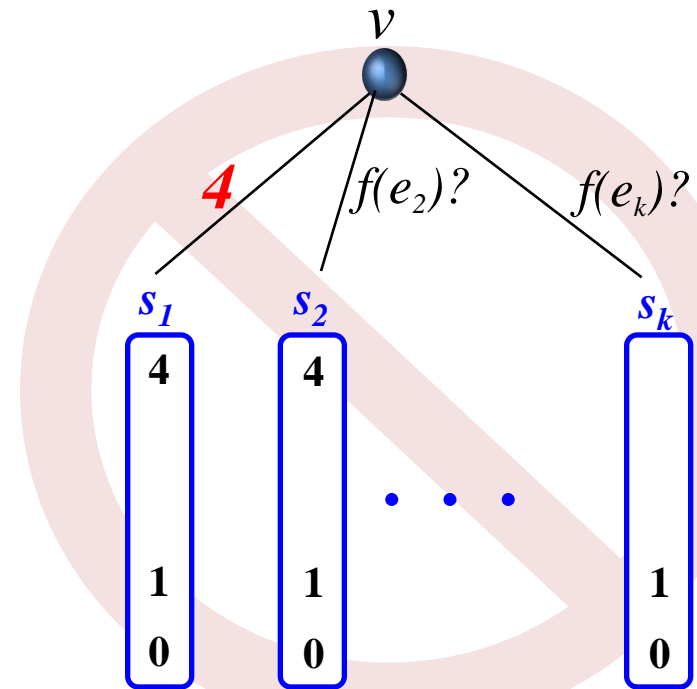


Edge-query model

● An extension assigns all $f(e_i)$'s

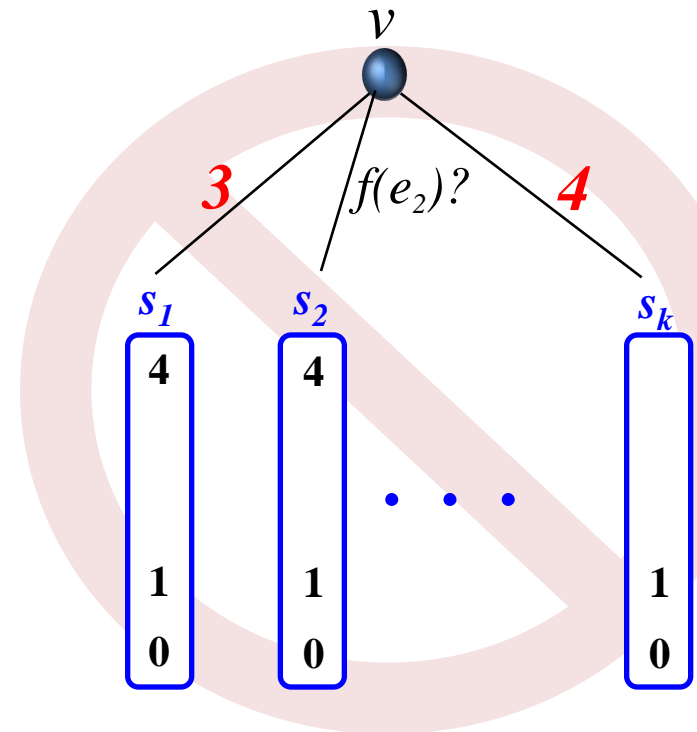
● $f(e_i) \neq f(e_j)$

● $f(e_i)$ is not in s_i



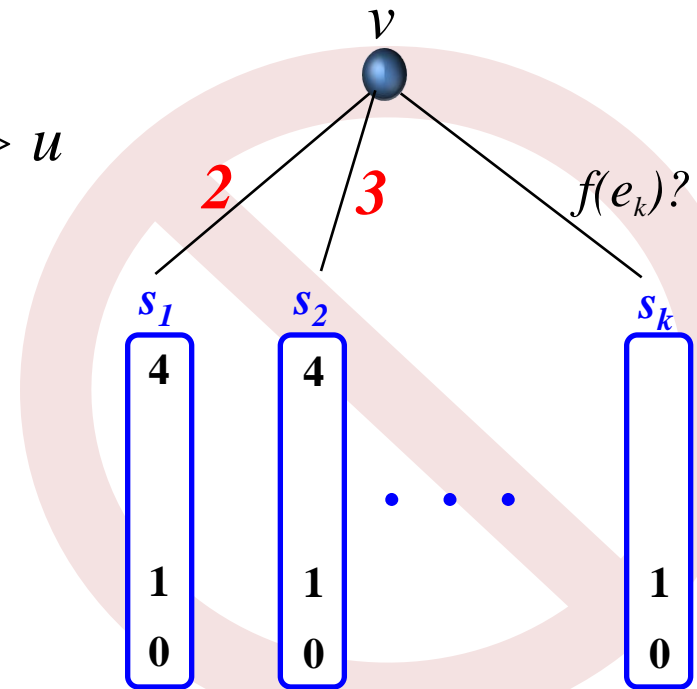
Edge-query model

- An extension assigns all $f(e_i)$'s
 - $f(e_i) \neq f(e_j)$
 - $f(e_i)$ is not in s_i
 - $f(e_i)$ is in $s_j \Rightarrow f(e_j) > f(e_i)$

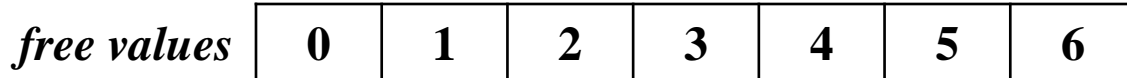
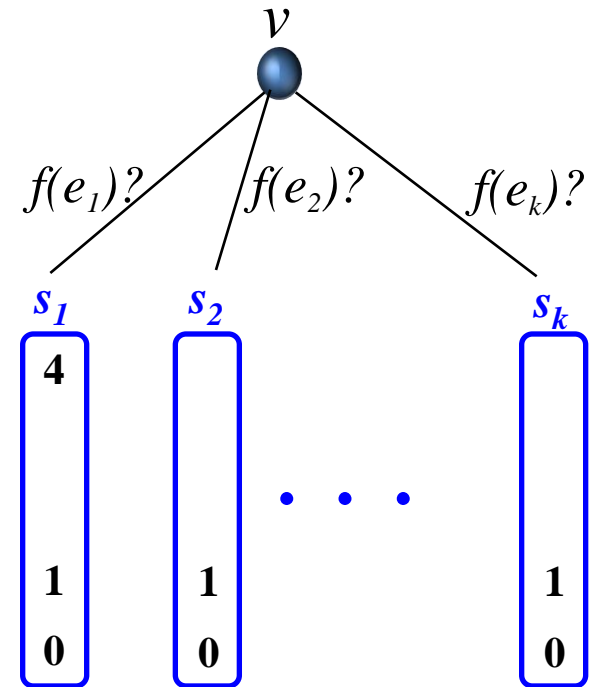


Edge-query model

- An extension assigns all $f(e_i)$'s
 - $f(e_i) \neq f(e_j)$
 - $f(e_i)$ is not in s_i
 - $f(e_i)$ is in $s_j \Rightarrow f(e_j) > f(e_i)$
 - u is in s_i and $s_j \Rightarrow \max\{f(e_i), f(e_j)\} > u$

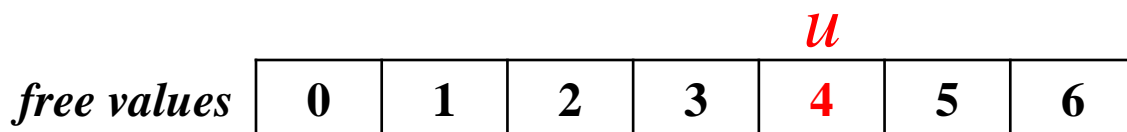
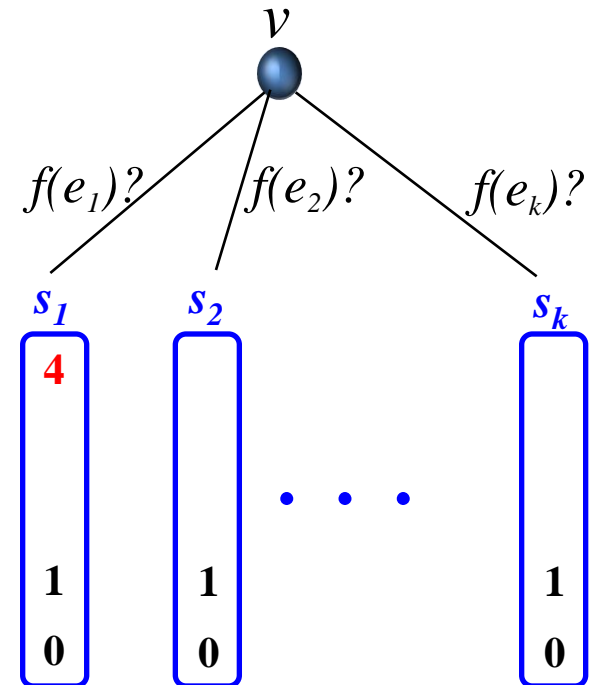


Algorithm Outline



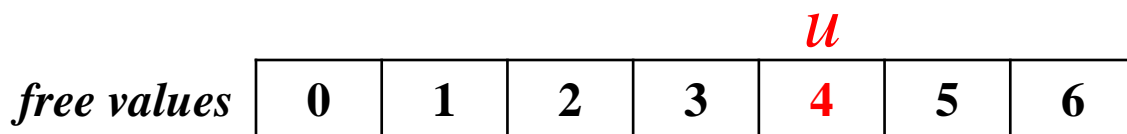
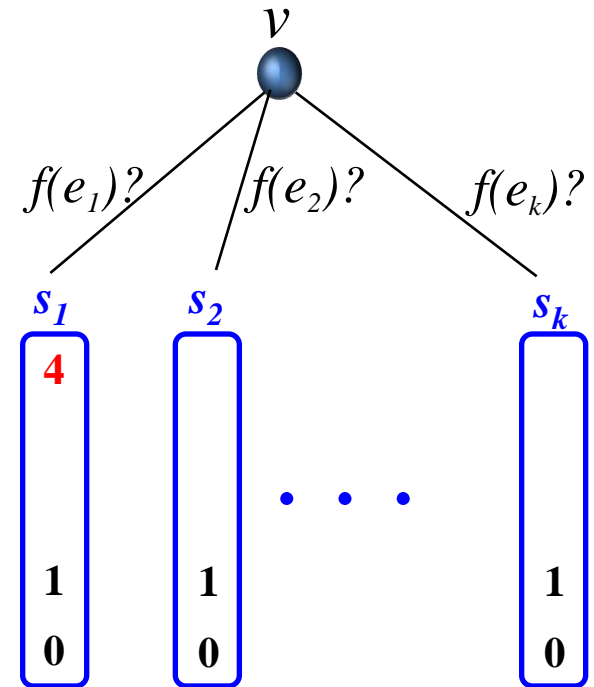
Algorithm Outline

● set $u = \max\{s_i\}$



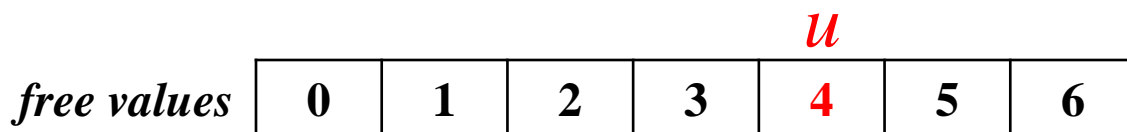
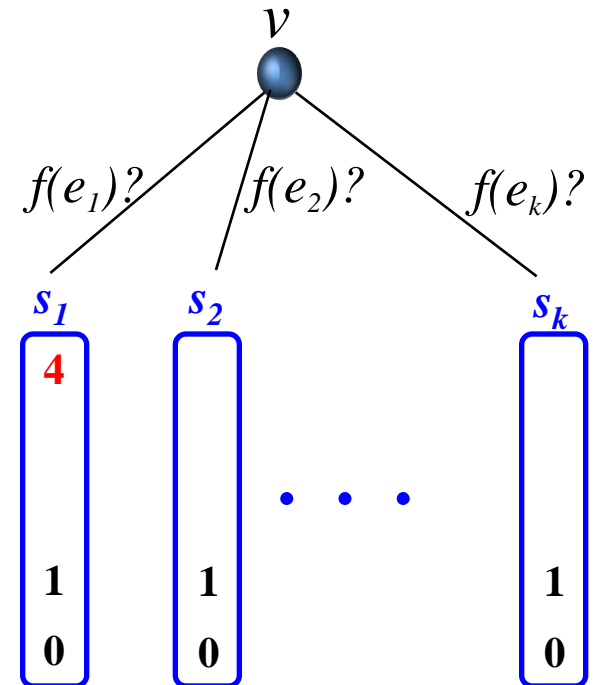
Algorithm Outline

- set $u = \max\{s_i\}$
- while not all edges assigned



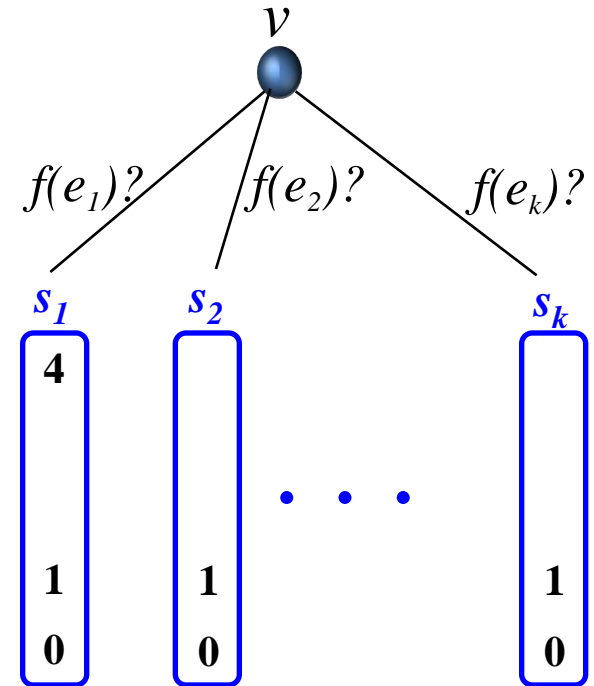
Algorithm Outline

- set $u = \max\{s_i\}$
- while not all edges assigned
 - if u appears once mark u as *not free*, move to next largest u



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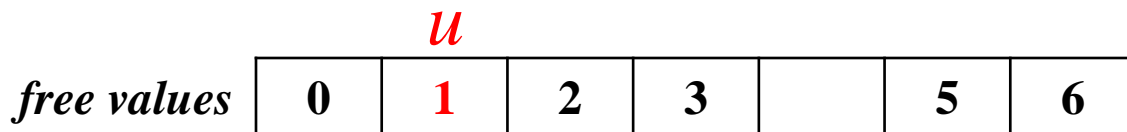
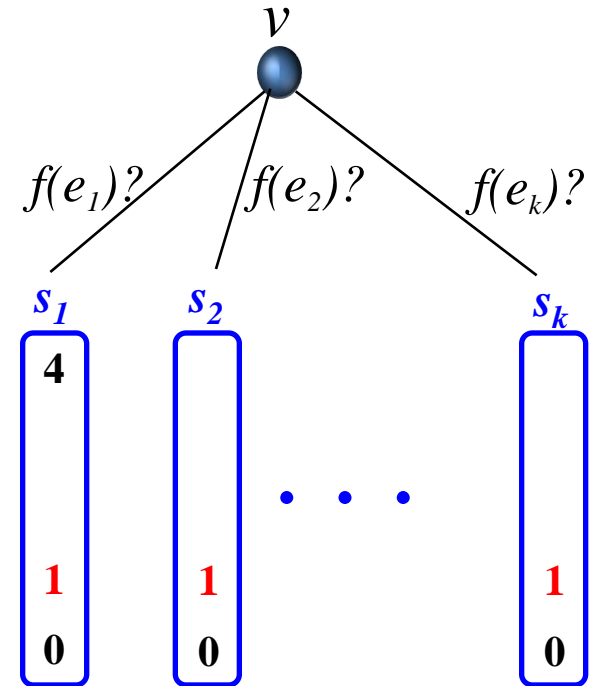


free values

0	1	2	3		5	6
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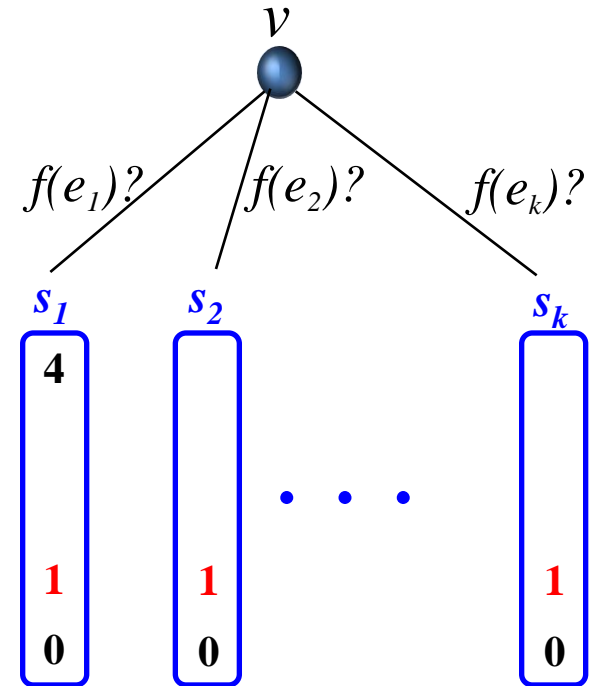
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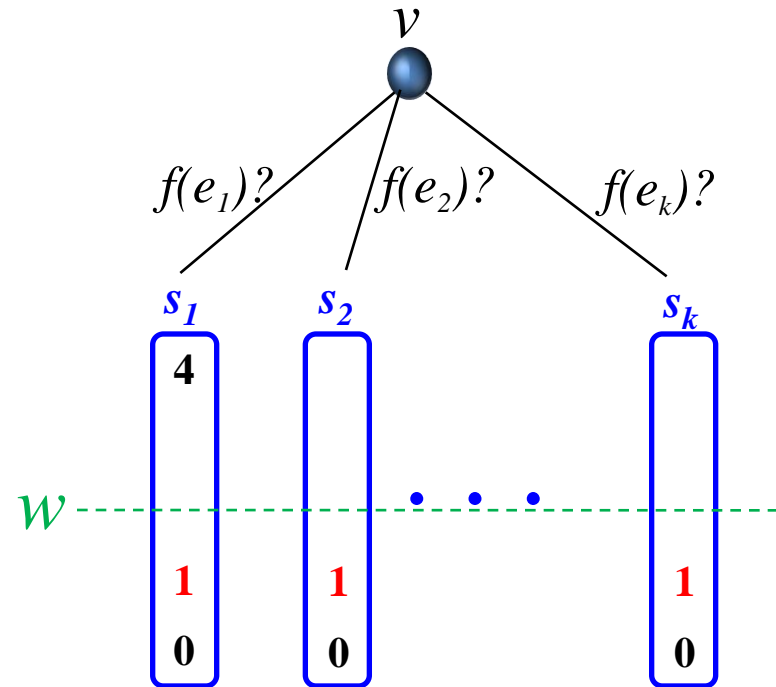


free values

		u	w			
0	1	2	3		5	6

Algorithm Outline

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 - if u appears once mark u as *not free*, move to next largest u
 - otherwise:
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 - $S_j =$ **any** maximal sequence w.r.t w

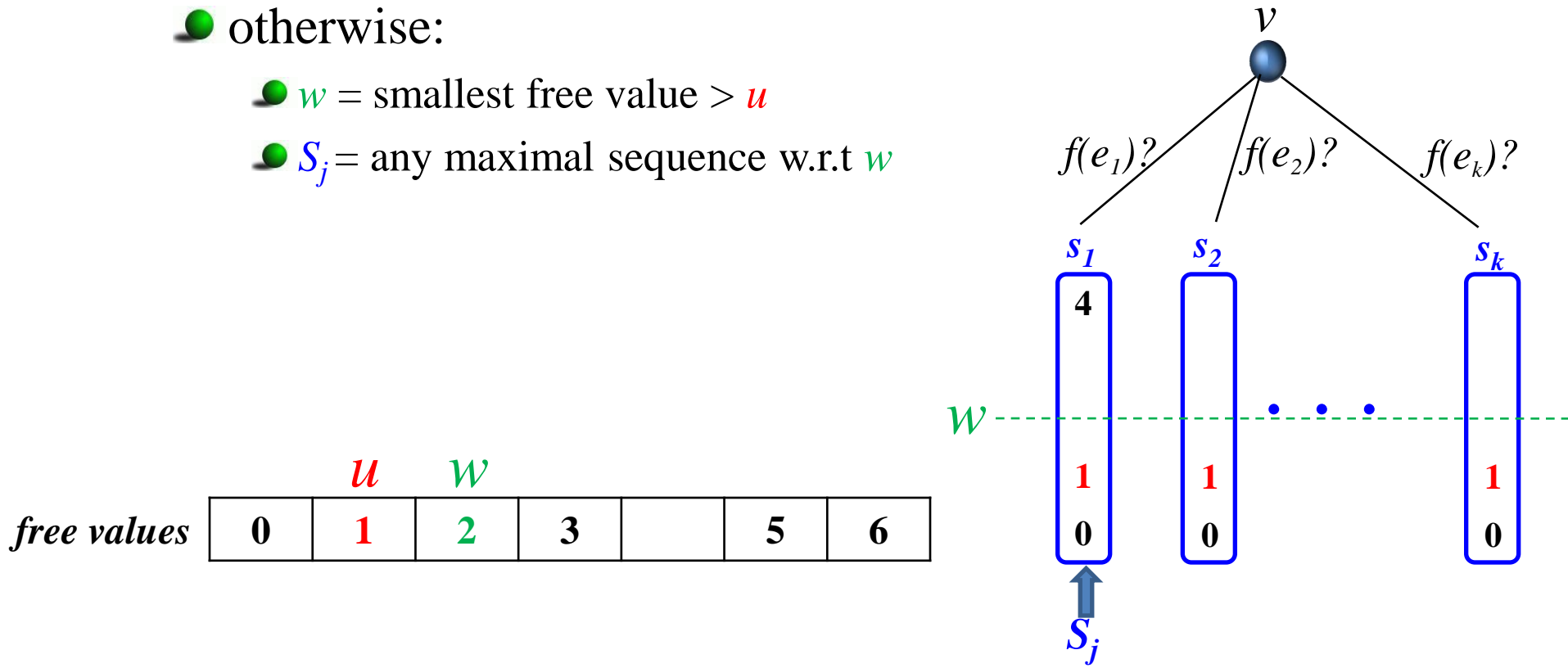


free values

	u	w				
0	1	2	3		5	6

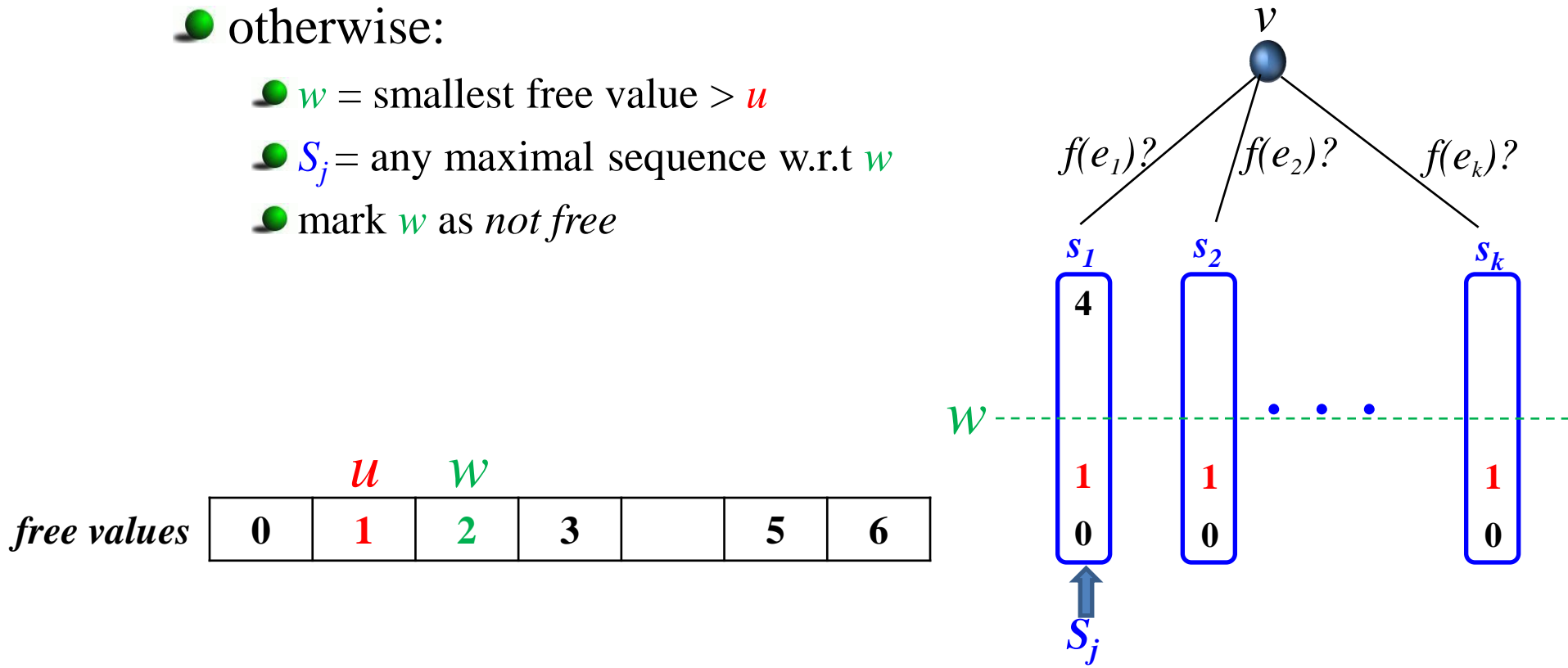
Algorithm Outline

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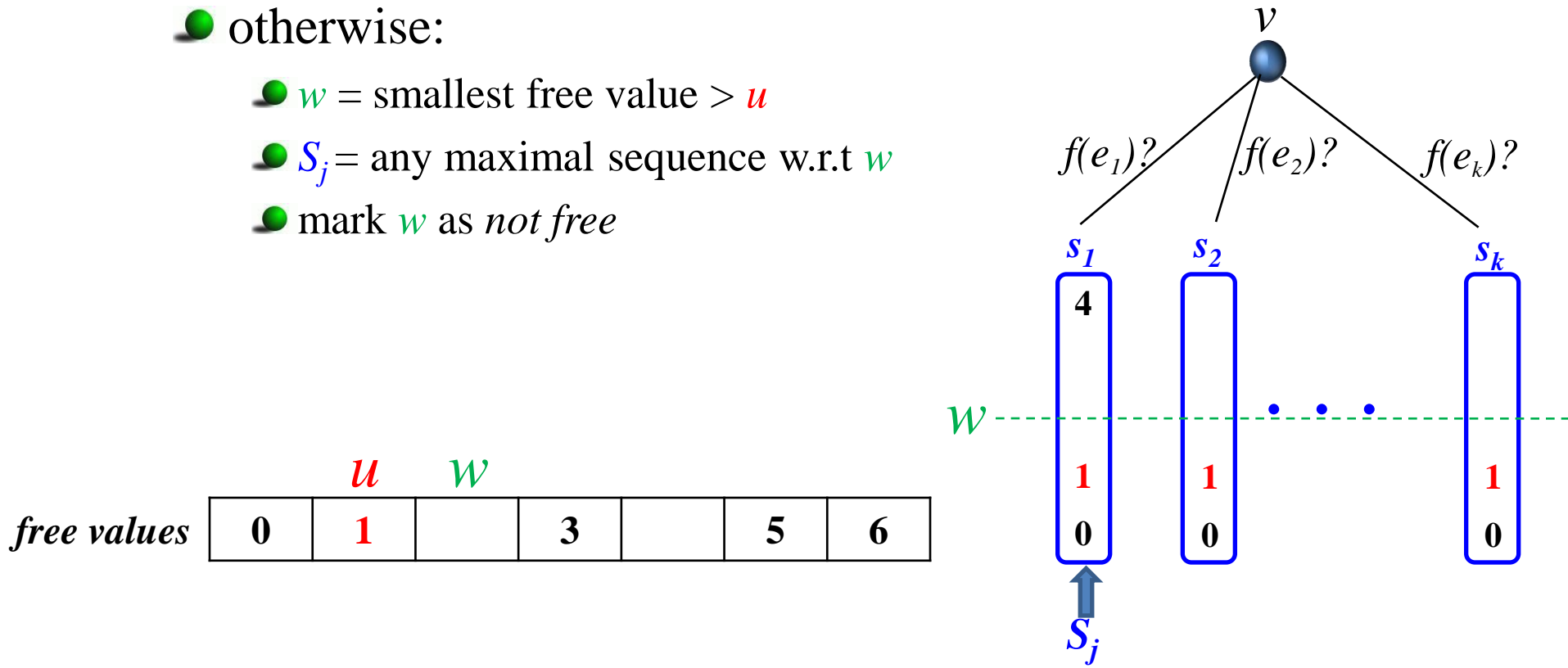
Algorithm Outline

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 - mark w as *not free*



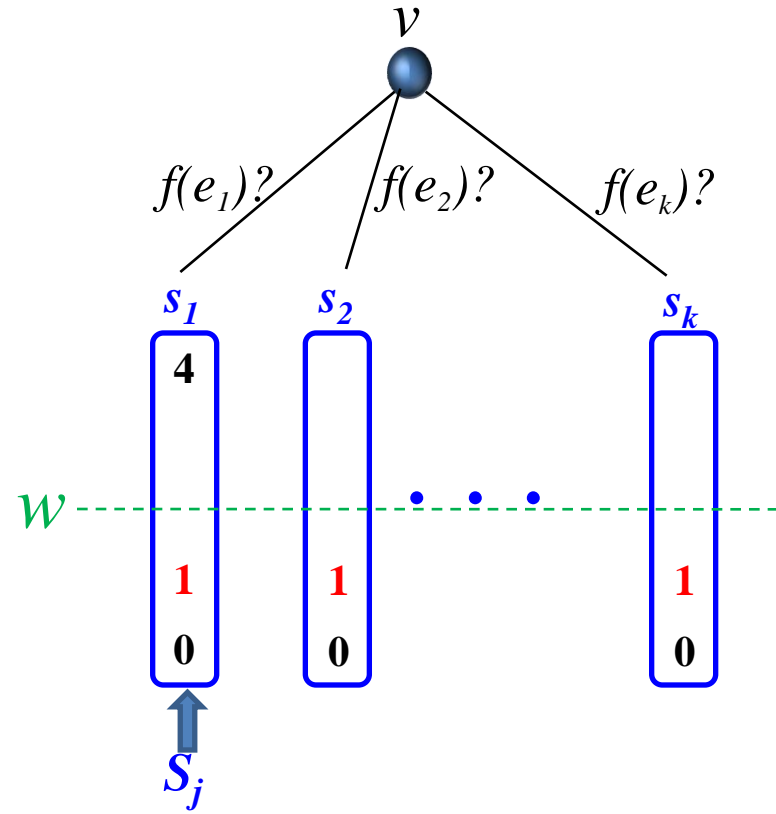
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 - mark w as *not free*
 - set **current** $f(e_j) = w$

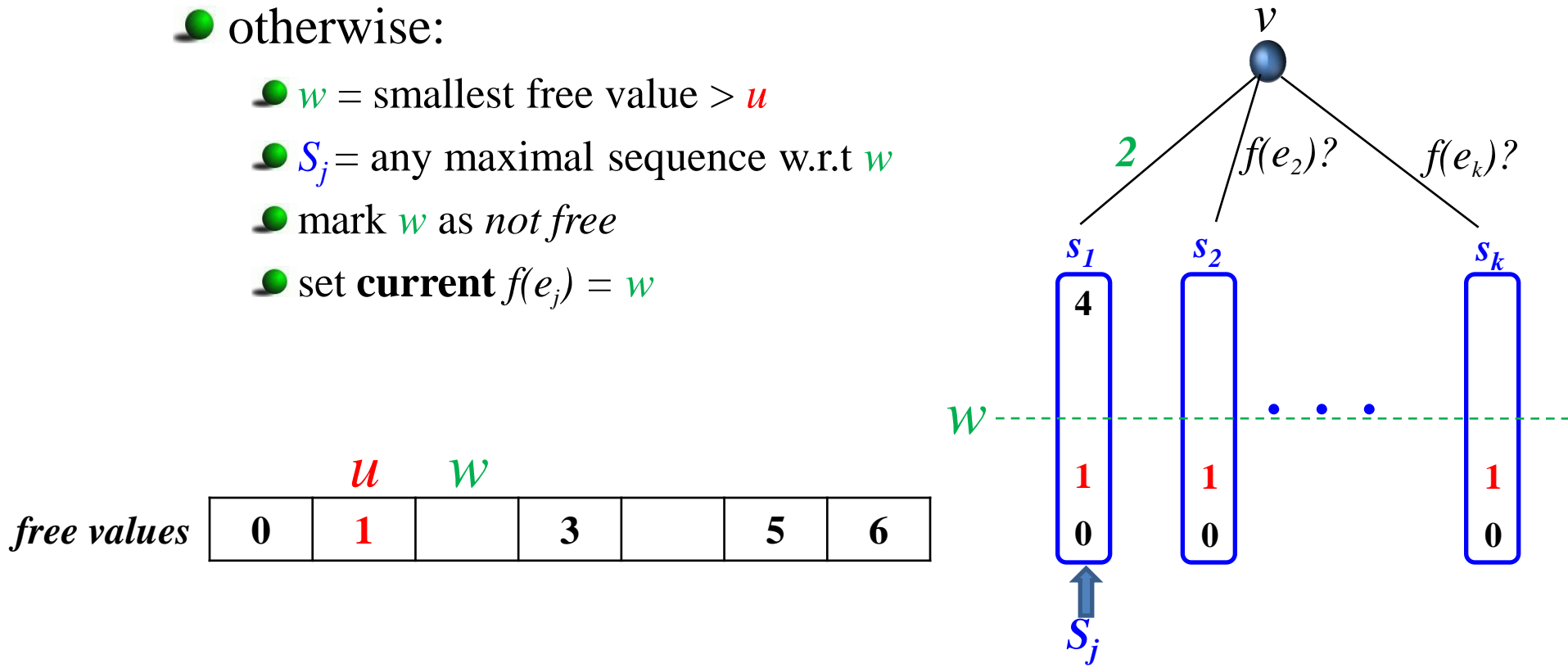


free values

	u	w				
0	1		3		5	6

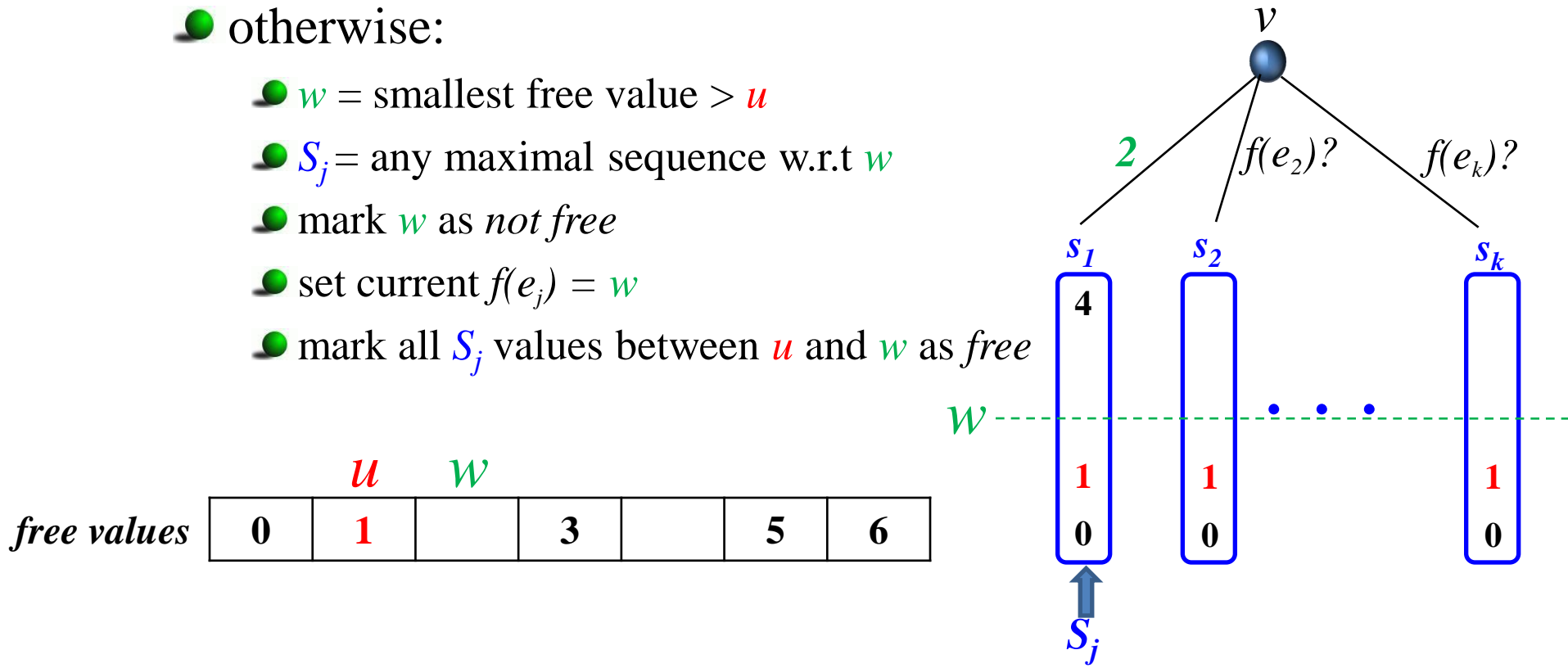
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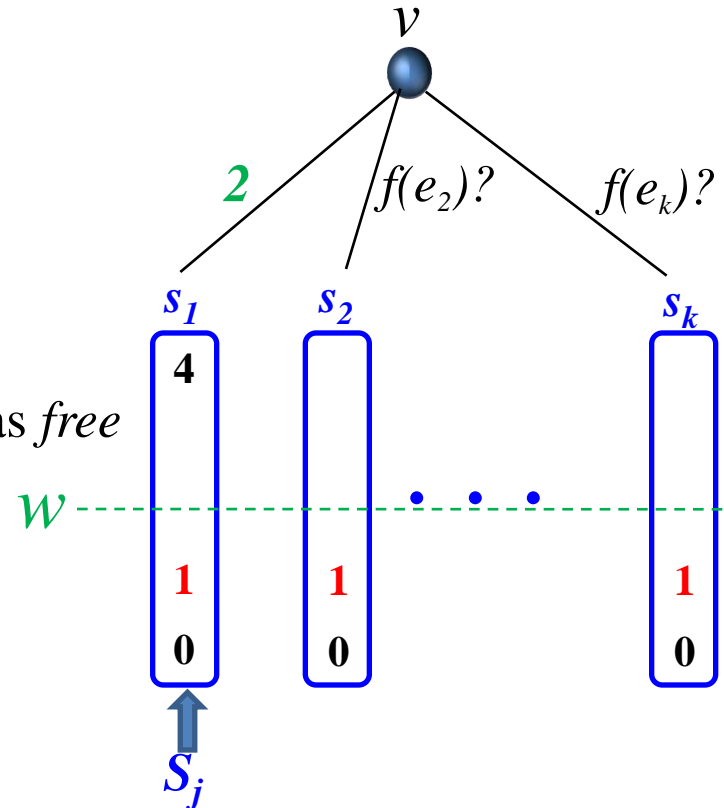
Algorithm Outline

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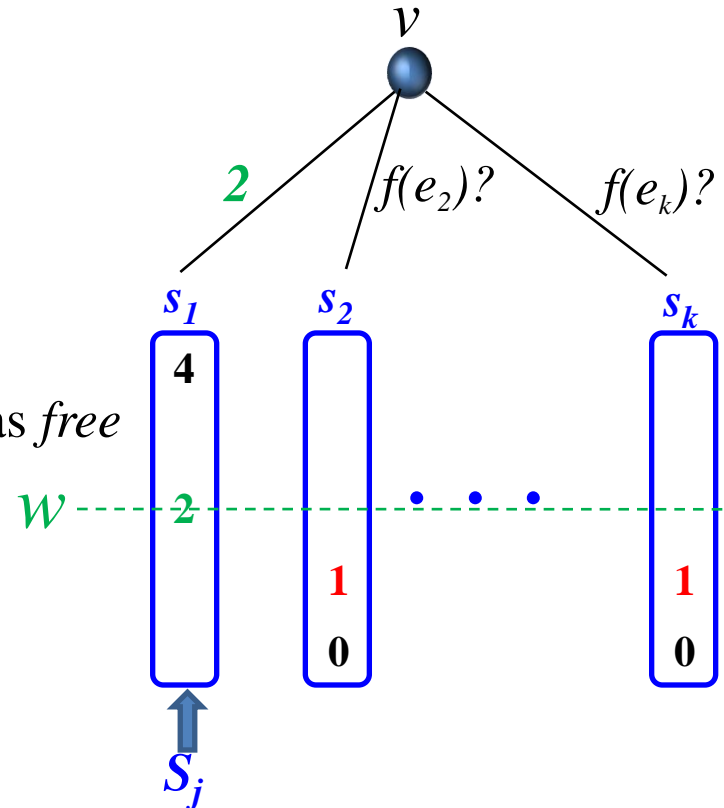


free values

	u	w				
0	1		3		5	6

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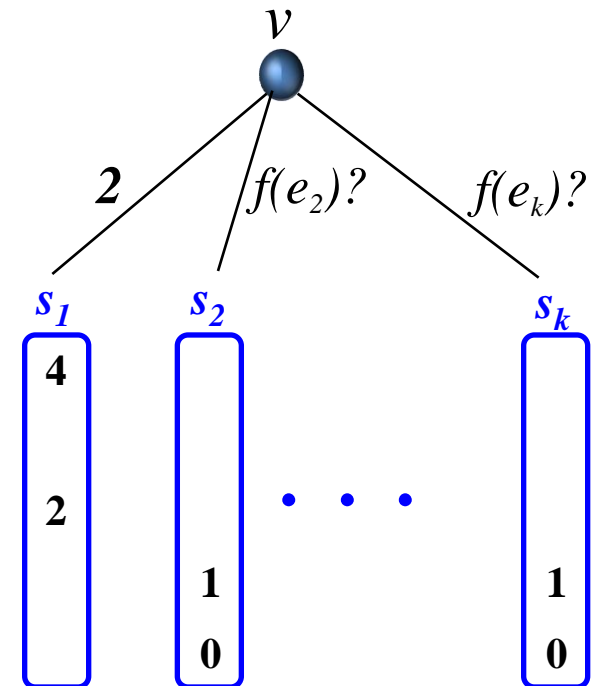


free values

	u		w				
0	1			3		5	6

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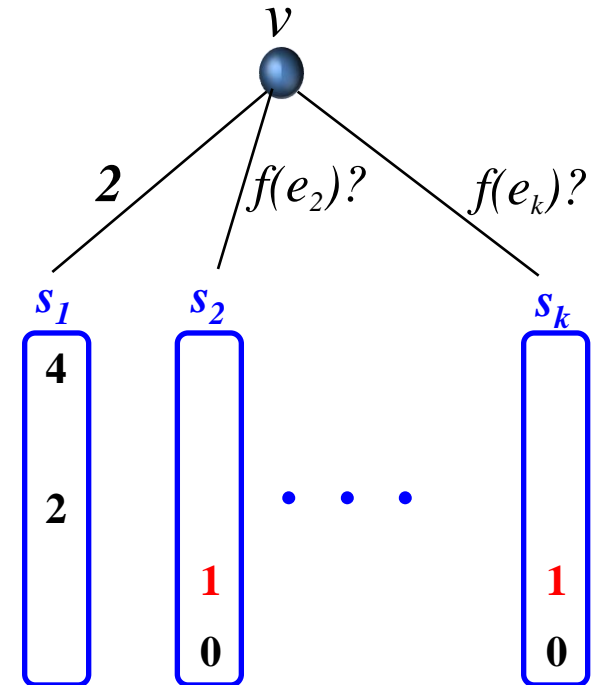


free values

0	1		3		5	6
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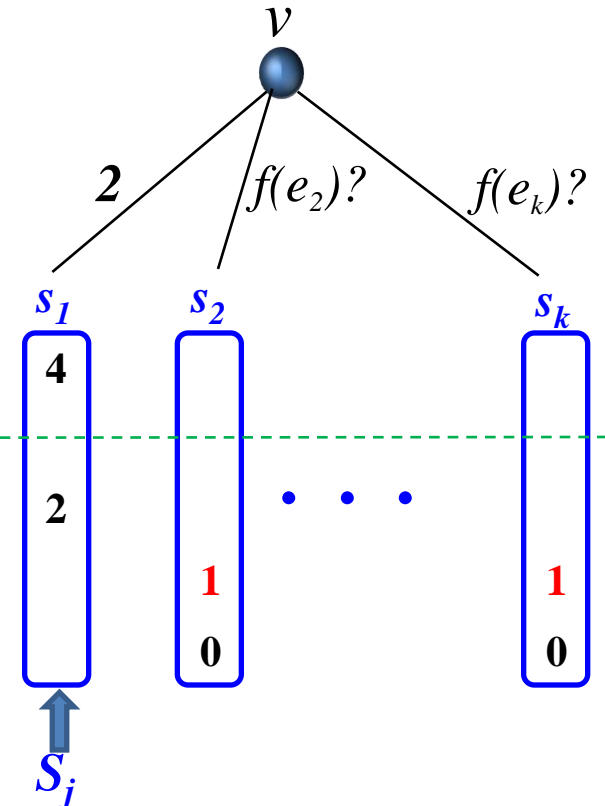


free values

	u					
0	1		3		5	6

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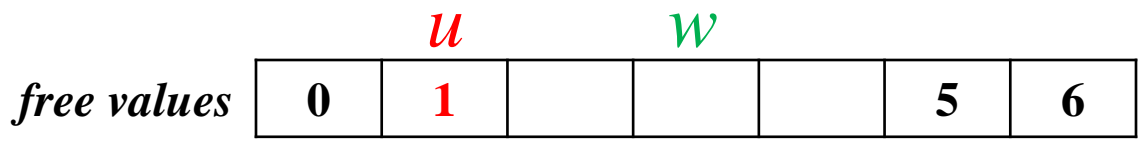
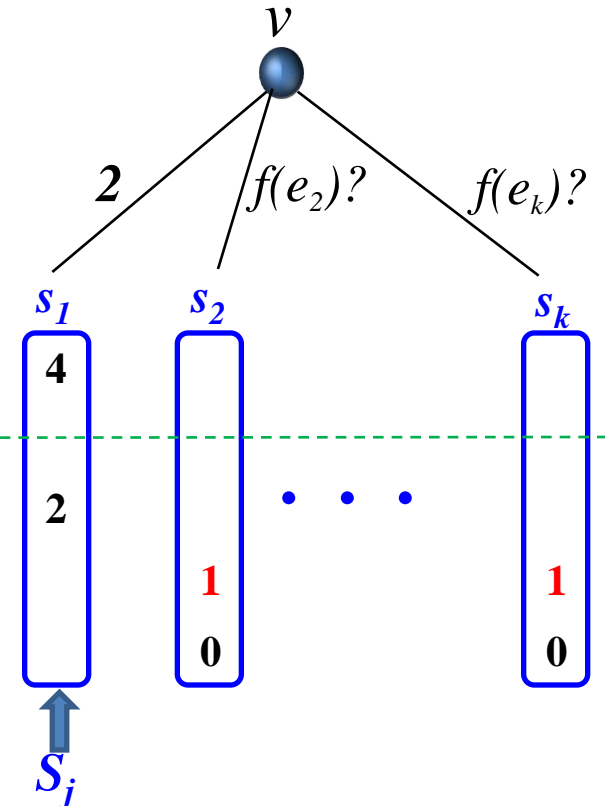


free values

	u		w			
0	1		3		5	6

Algorithm Outline

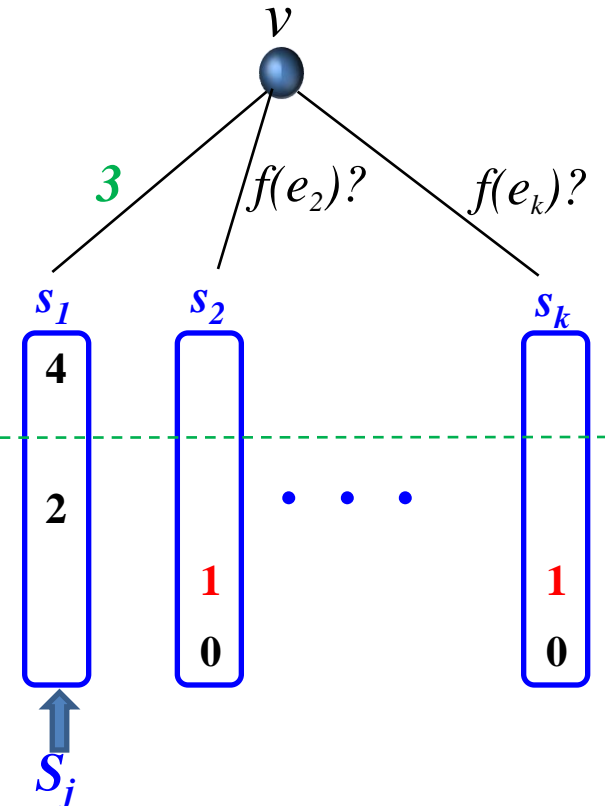
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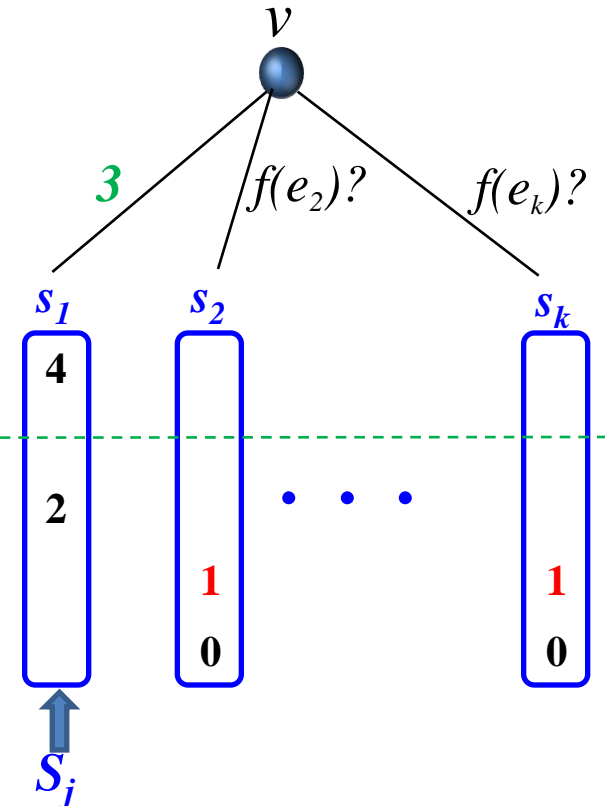


free values

	u						
0	1				5	6	

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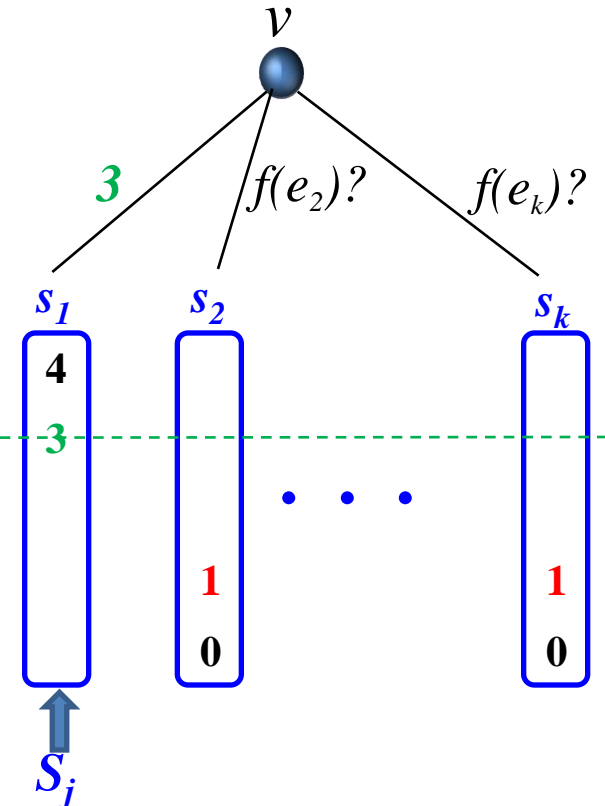


free values

	u		w			
0	1	2			5	6

Algorithm Outline

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- while not all edges assigned
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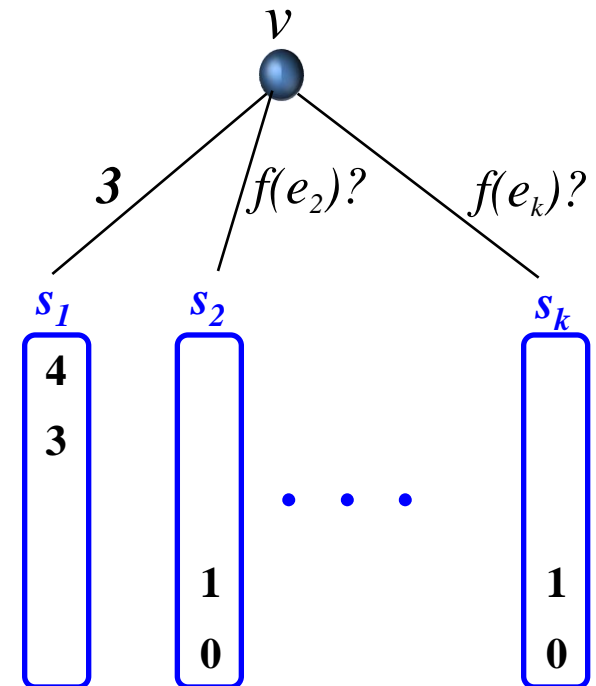


free values

	u		w			
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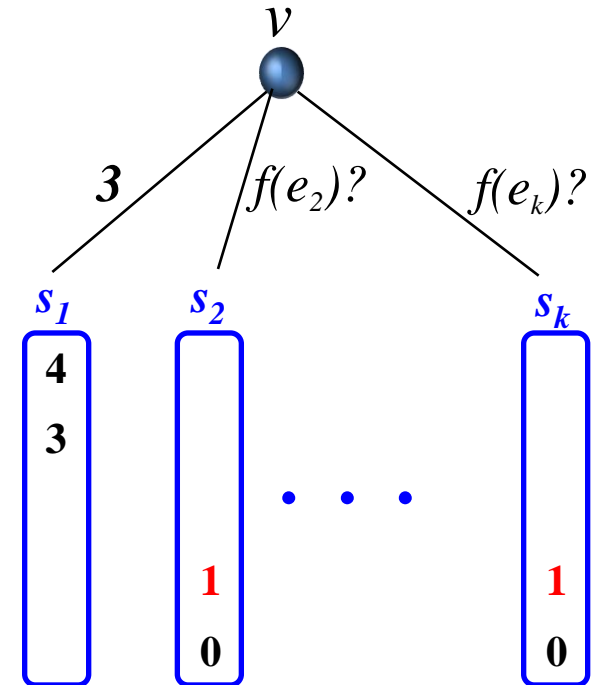


free values

0	1	2			5	6
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u

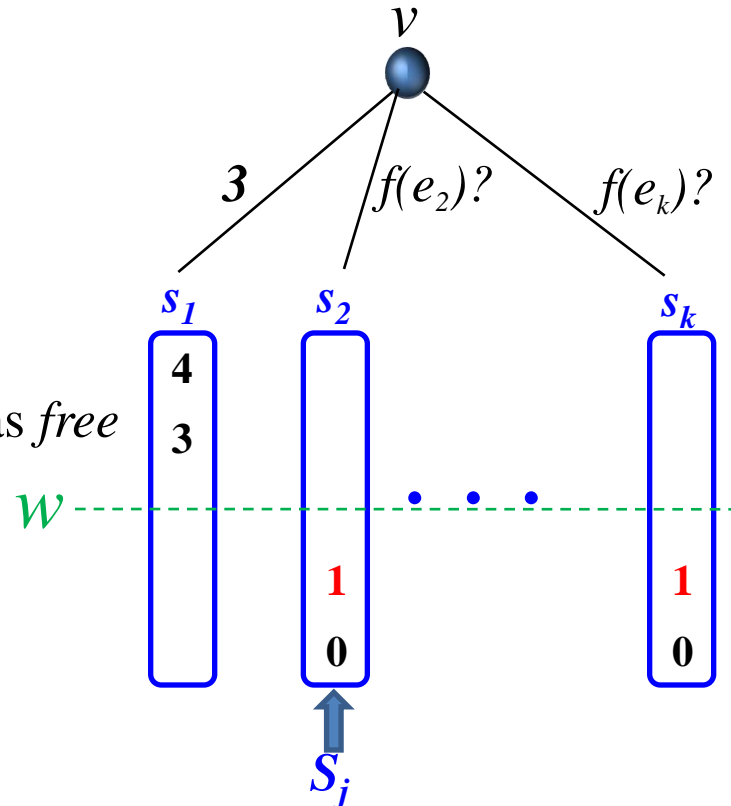
free values

0	1	2			5	6
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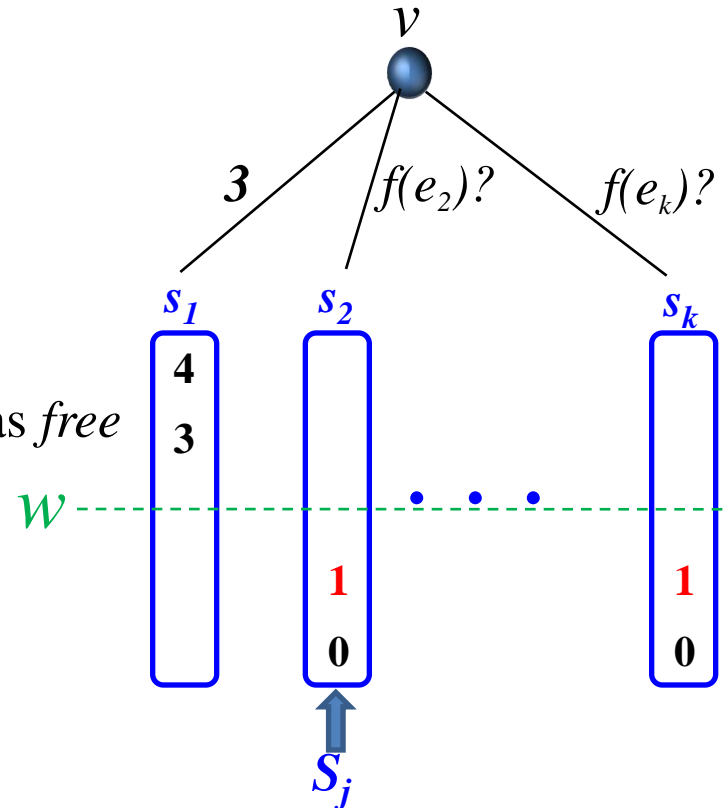


free values

	u	w				
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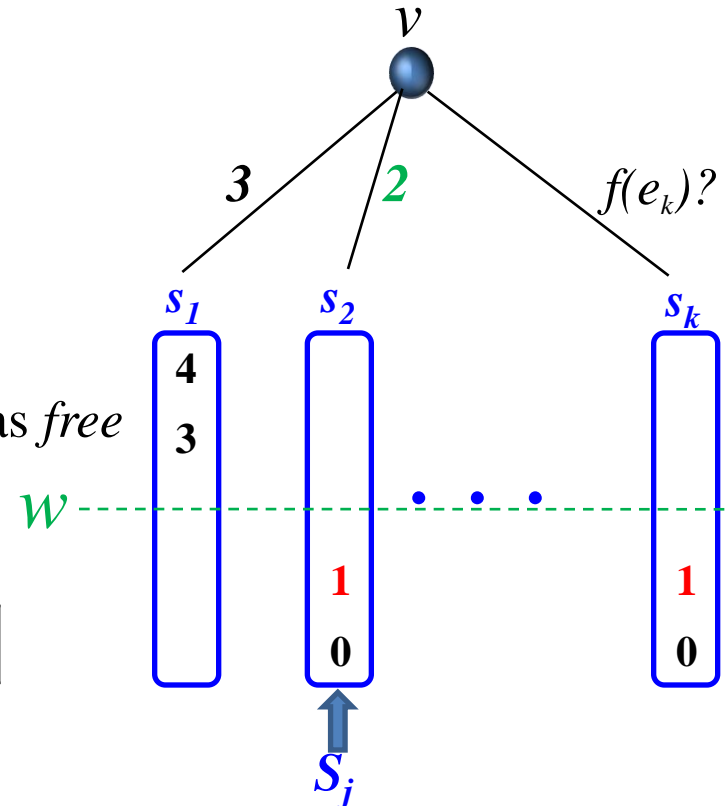
free values

	u		w				
0	1				5	6	

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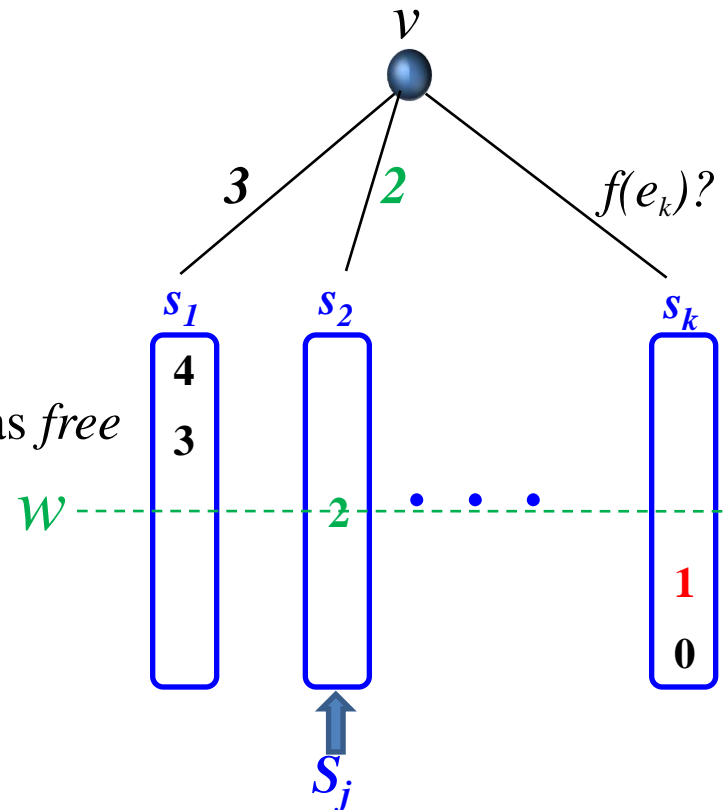


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	u		w				
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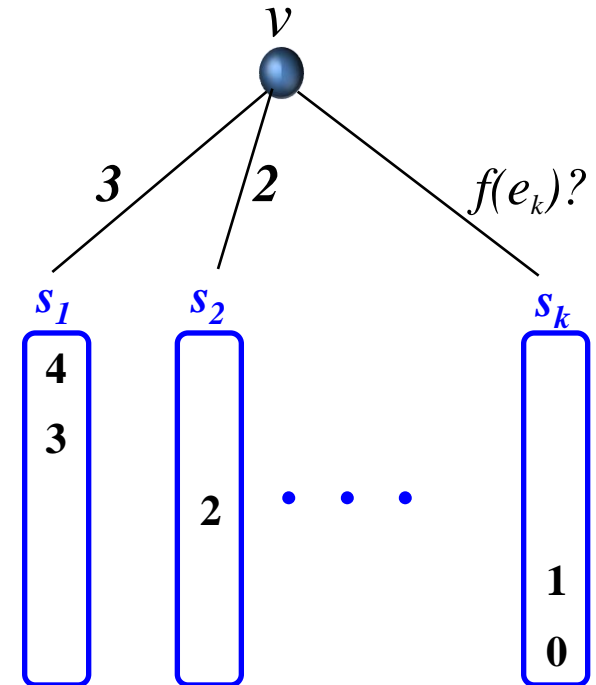


free values

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0	1				5	6	

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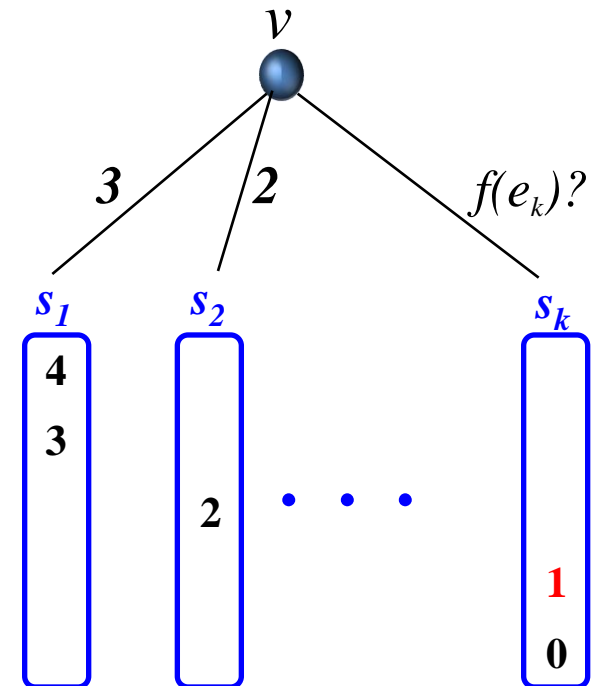


free values

0	1				5	6
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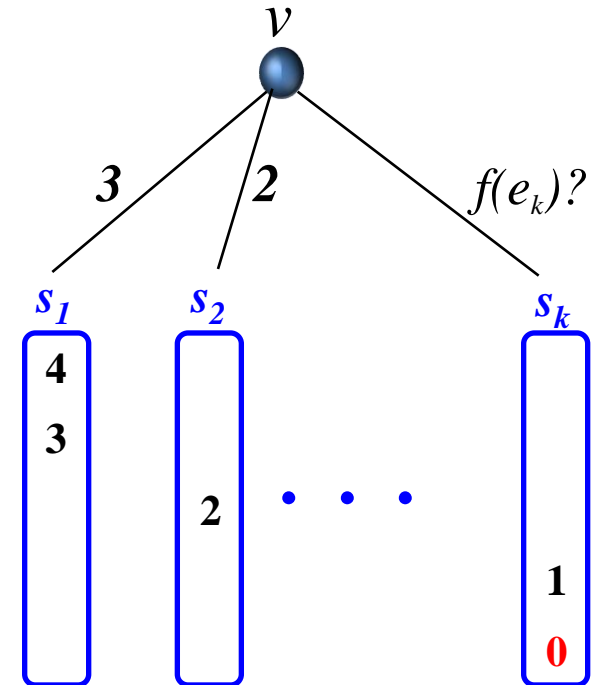
free values

	u					
0	1				5	6

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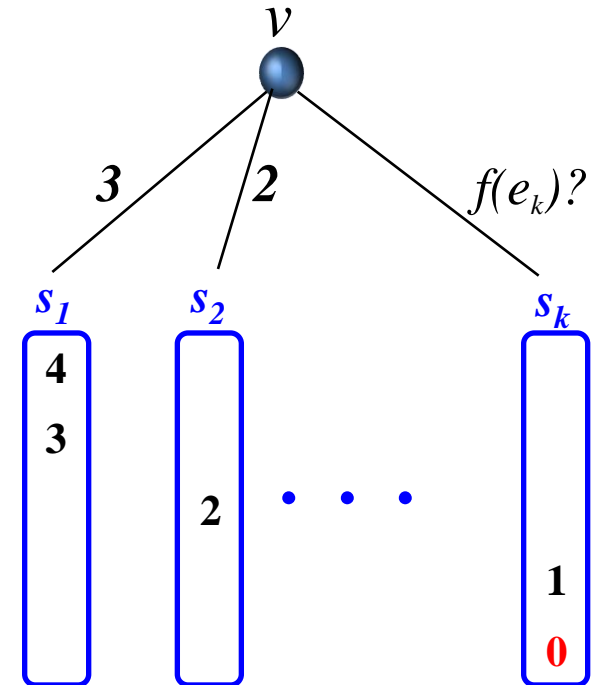
u

free values

0					5	6
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Algorithm Outline

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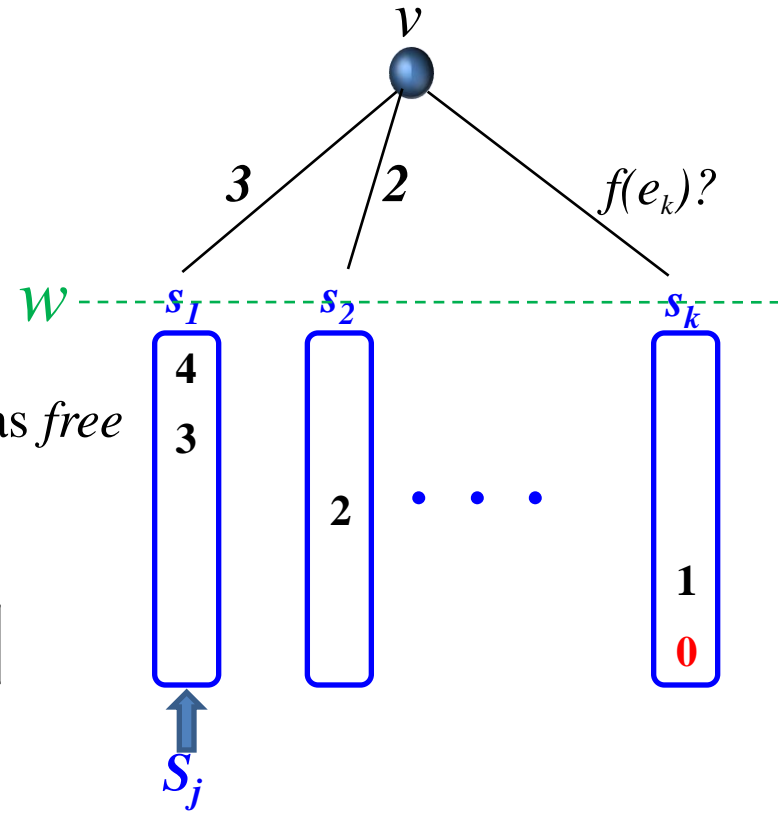


free values

u							
0					5	6	

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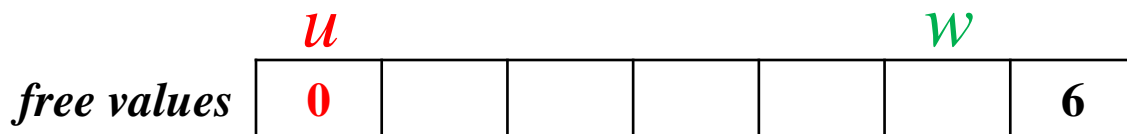
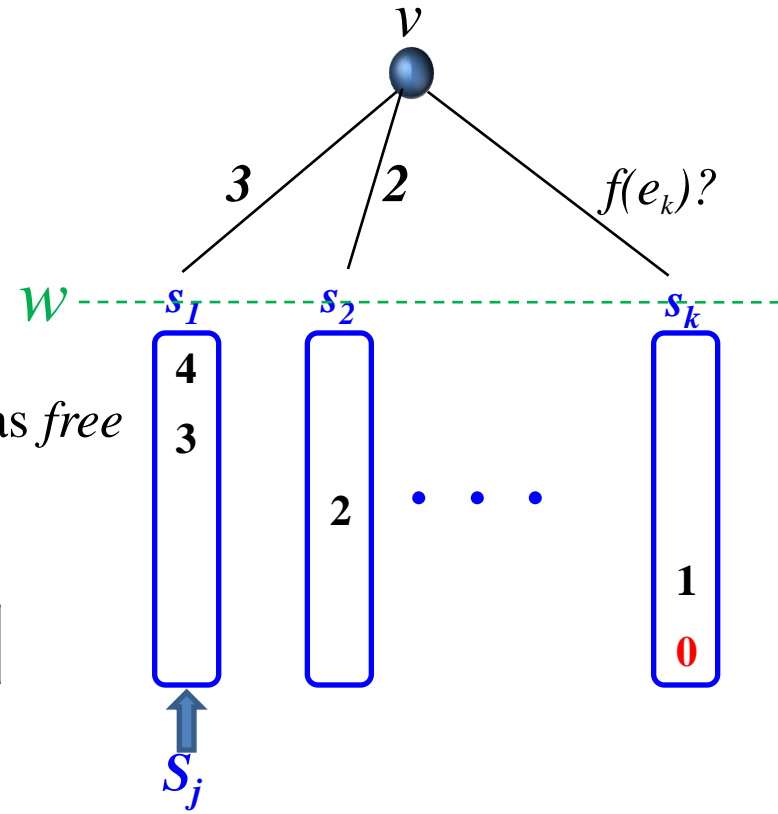


free values

u					w	
0					5	6

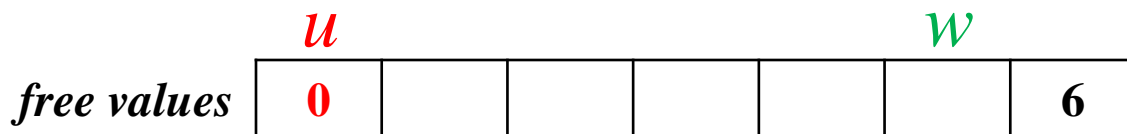
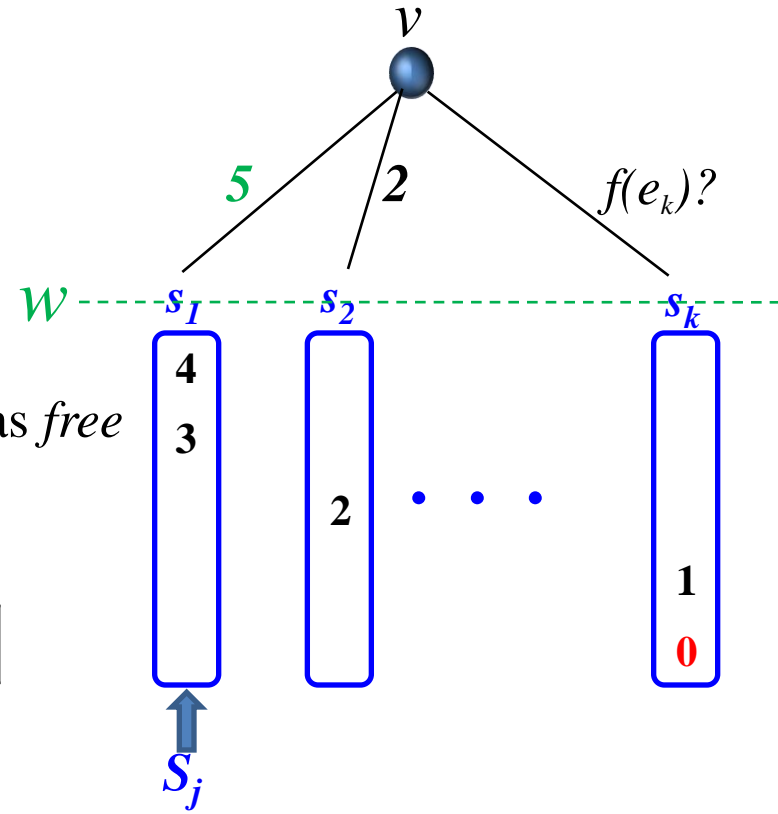
Algorithm Outline

- set $u = \max\{s_i\}$
- while not all edges assigned
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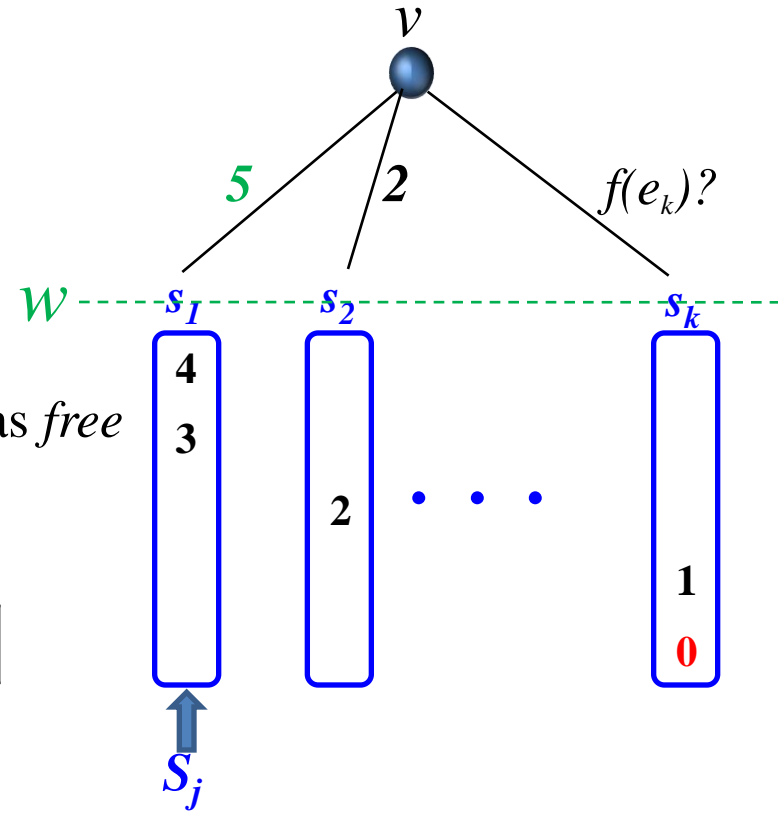
Algorithm Outline

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Algorithm Outline

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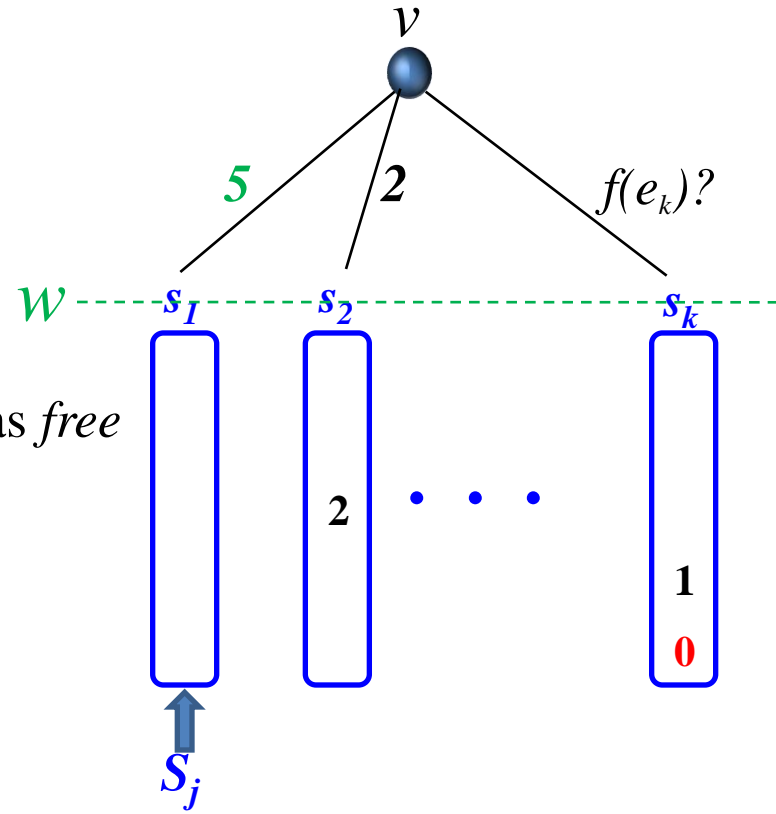


free values

u					w	
0			3	4		6

Algorithm Outline

- set $u = \max\{s_i\}$
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 - otherwise: and $u \neq 0$
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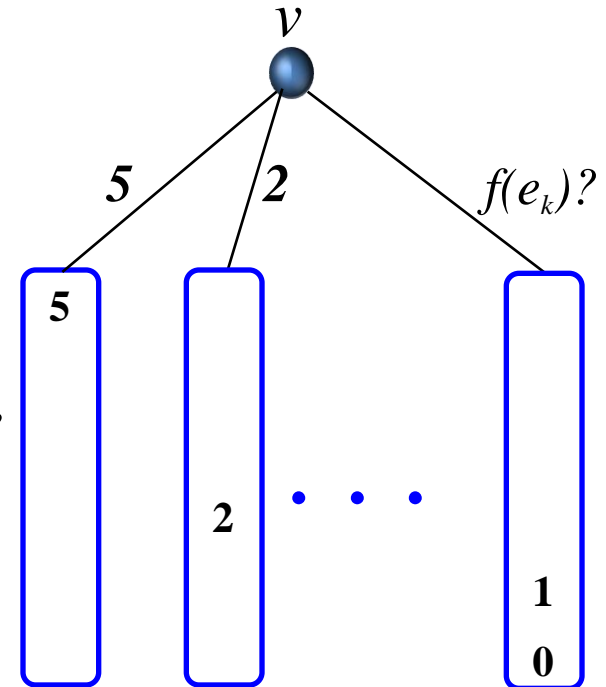


free values

u					w	
0			3	4		6

Algorithm Outline

- set $u = \max\{s_i\}$
- while not all edges assigned
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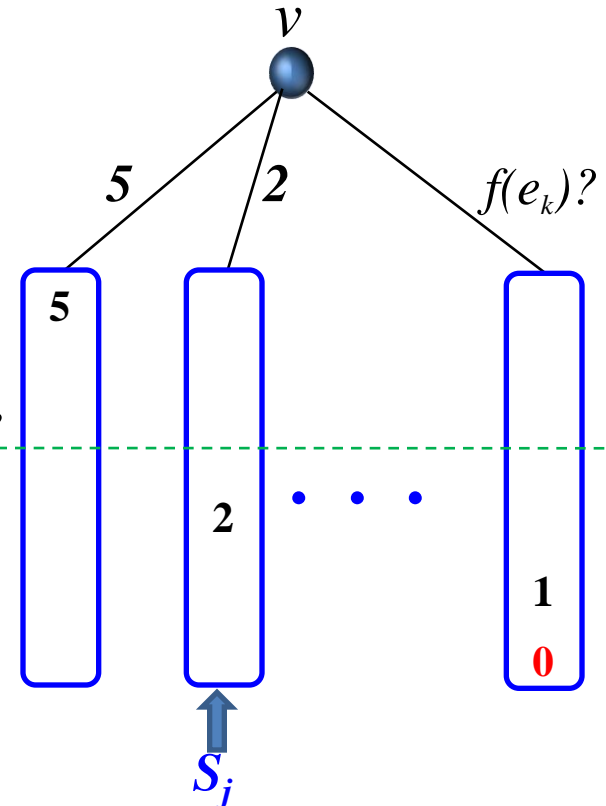
free values

0			3	4		6
---	--	--	---	---	--	---

Algorithm Outline

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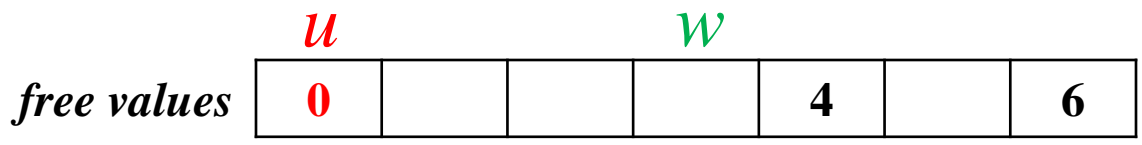
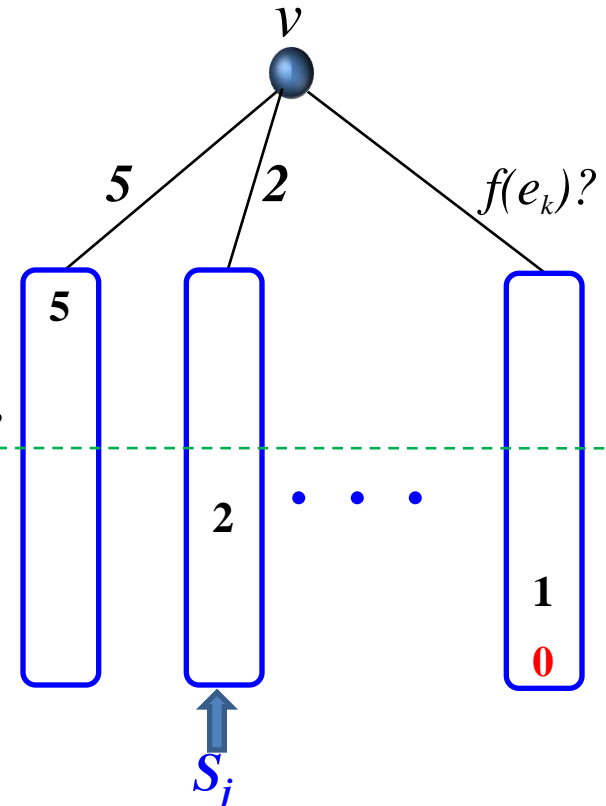


free values

u			w			
0			3	4		6

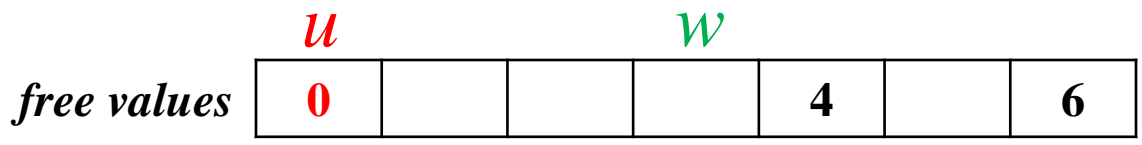
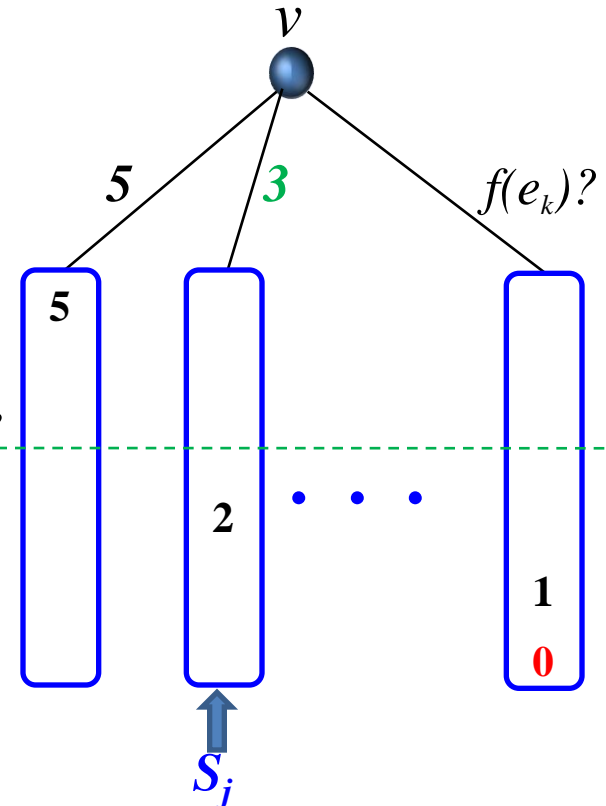
Algorithm Outline

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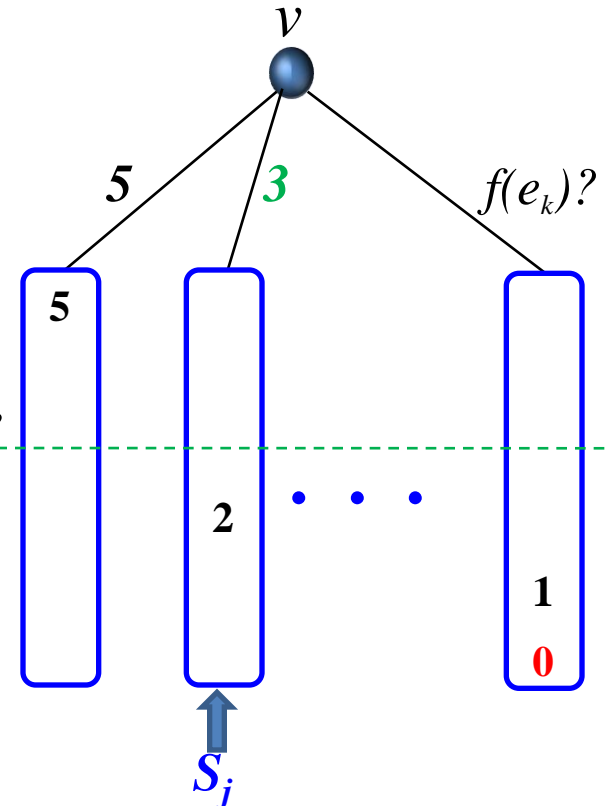
Algorithm Outline

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Algorithm Outline

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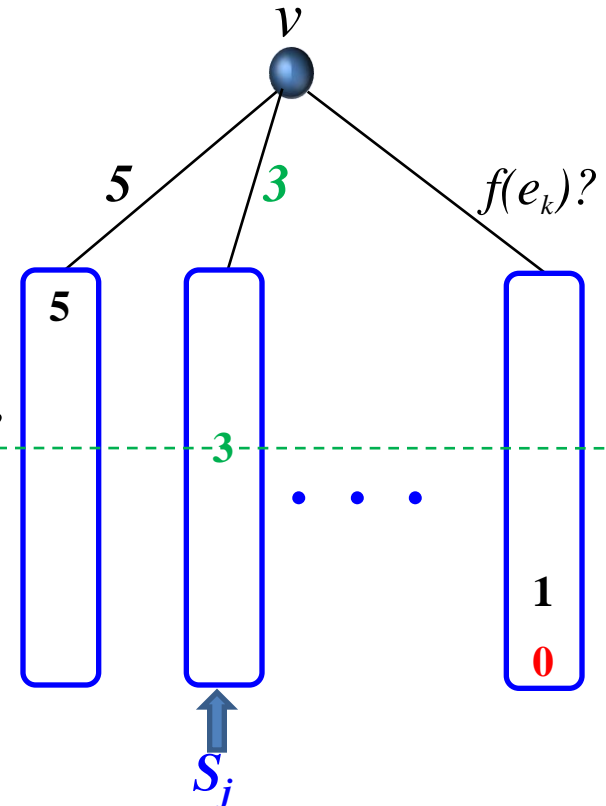


free values

u			w			
0		2		4		6

Algorithm Outline

- set $u = \max\{s_i\}$
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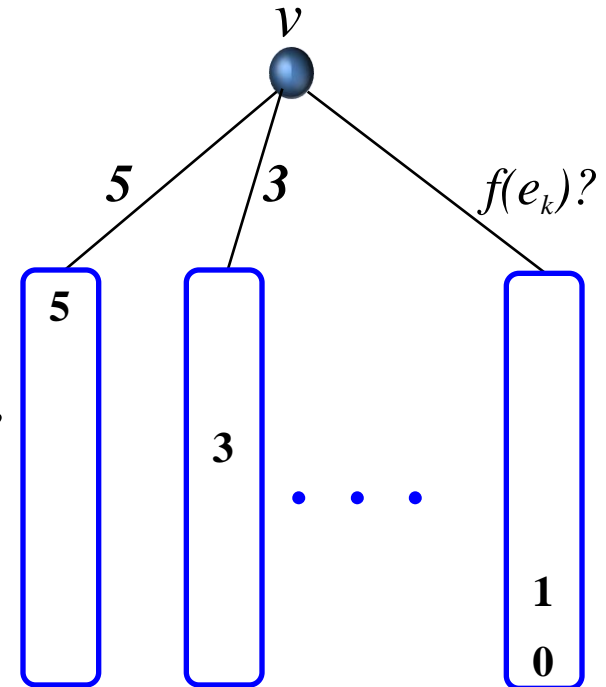


free values

u			w			
0		2		4		6

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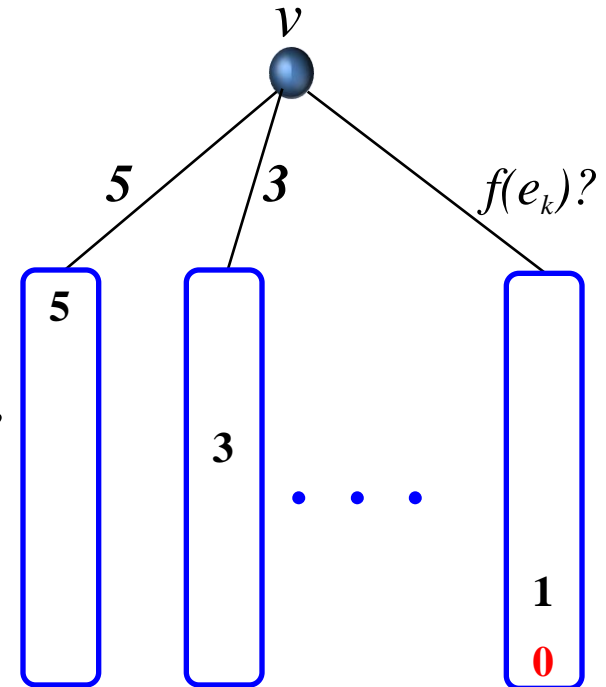


free values

0		2		4		6
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Algorithm Outline

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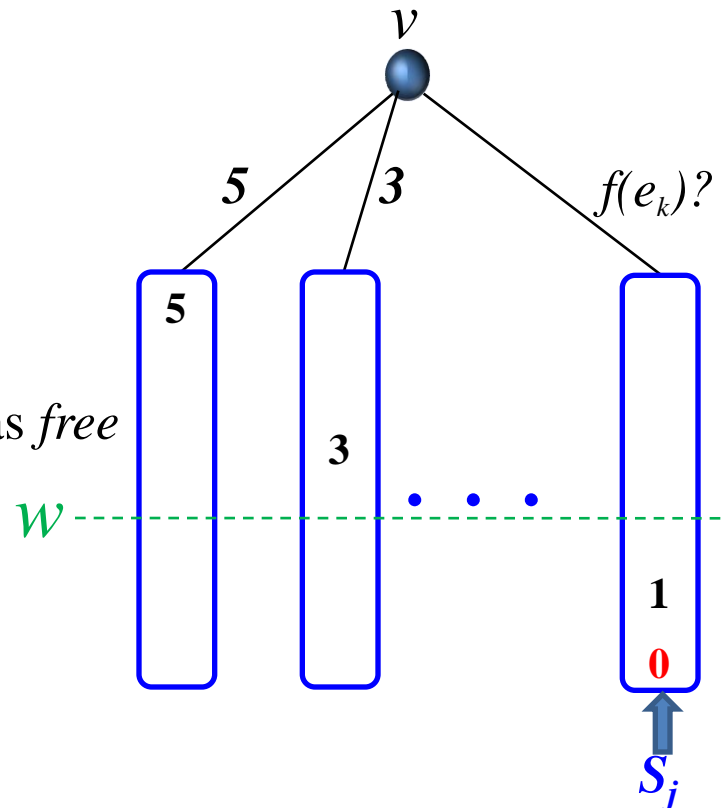
free values

u						
0		2		4		6

Algorithm Outline

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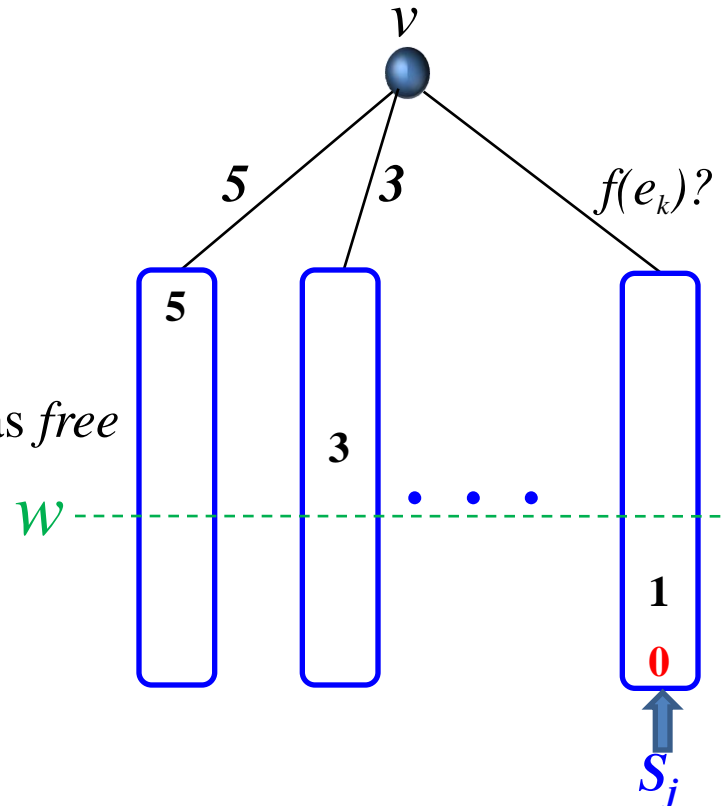
free values

u		w				
0		2		4		6

Algorithm Outline

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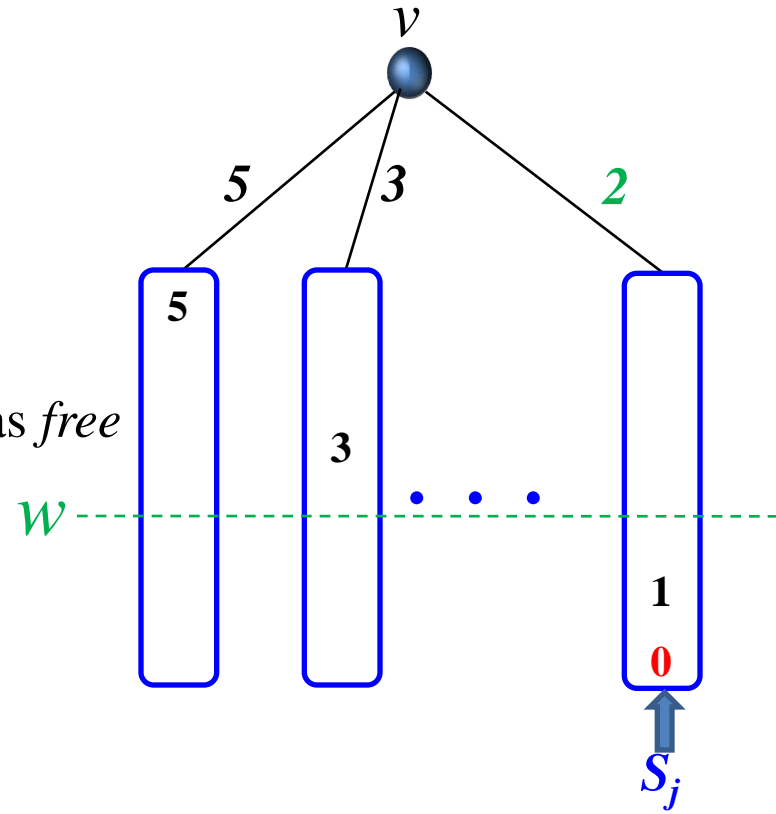


free values

u		w				
0				4		6

Algorithm Outline

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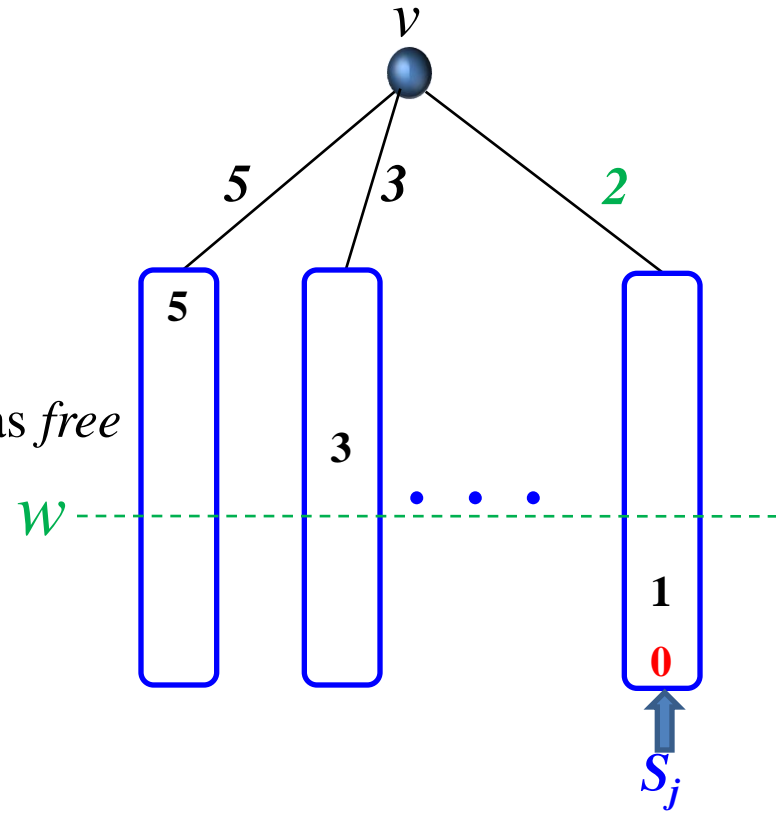


free values

u		w					
0					4		6

Algorithm Outline

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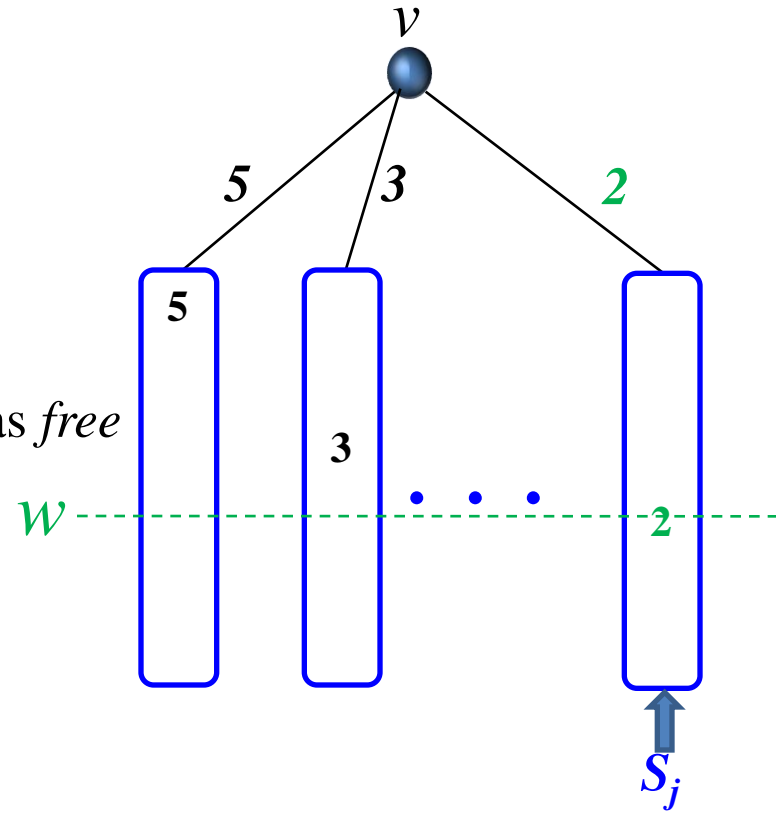


free values

u	w				
0	1			4	6

Algorithm Outline

- set $u = \max\{s_i\}$
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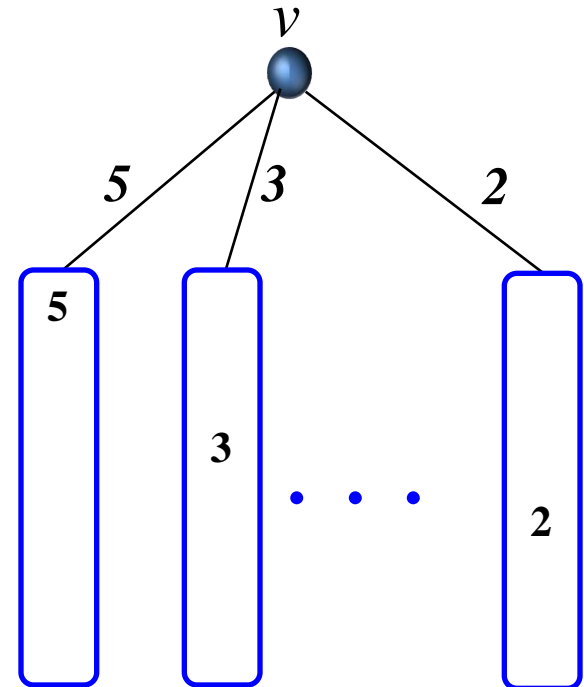
free values

u		w				
0	1			4		6

Algorithm Outline

That's it!

- set $u = \max\{s_i\}$
- while not all edges assigned
 - if u appears once, mark u as *not free*, move to next largest u
 - otherwise: $\overbrace{\text{and } u \neq 0}$
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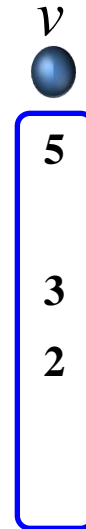
free values

0	1			4		6
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Algorithm Outline

That's it!

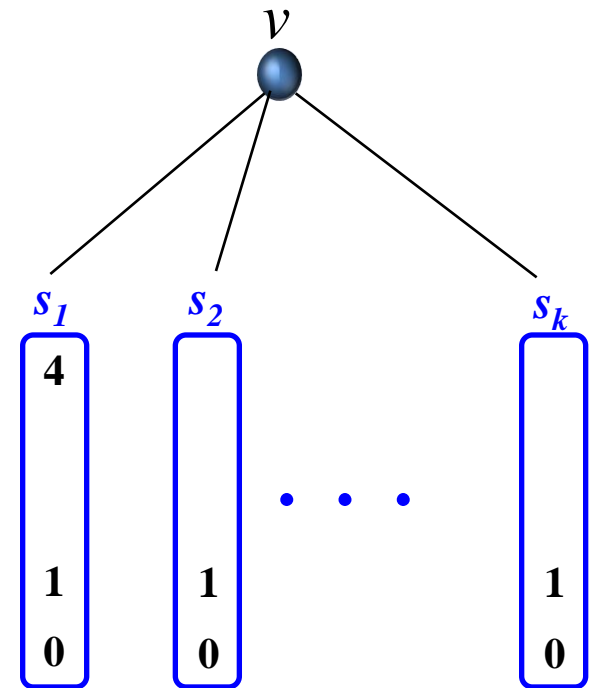
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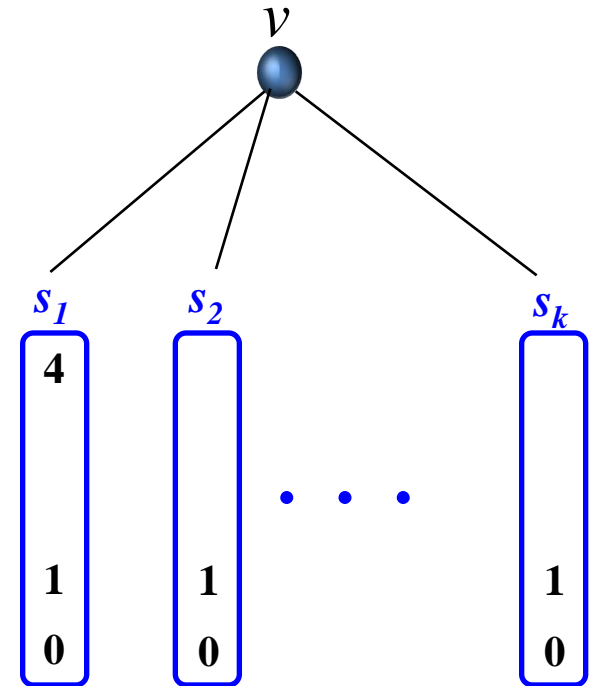
0	1			4		6
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Running Time



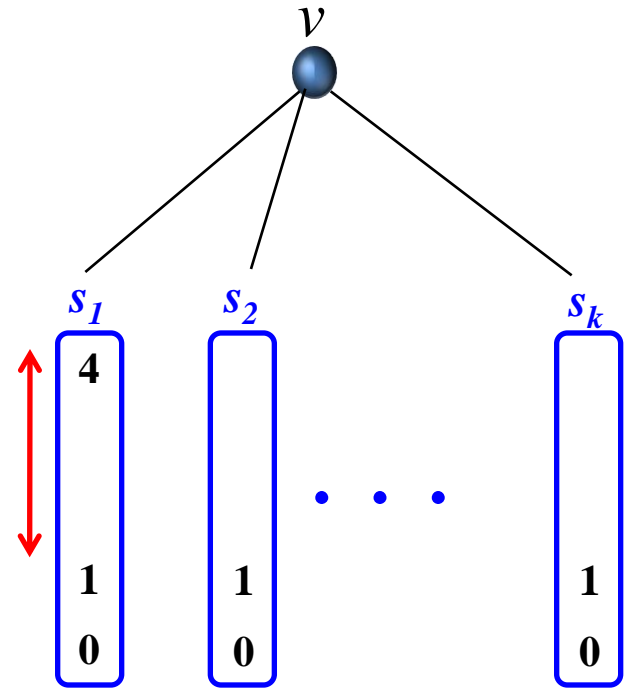
Running Time

• $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !



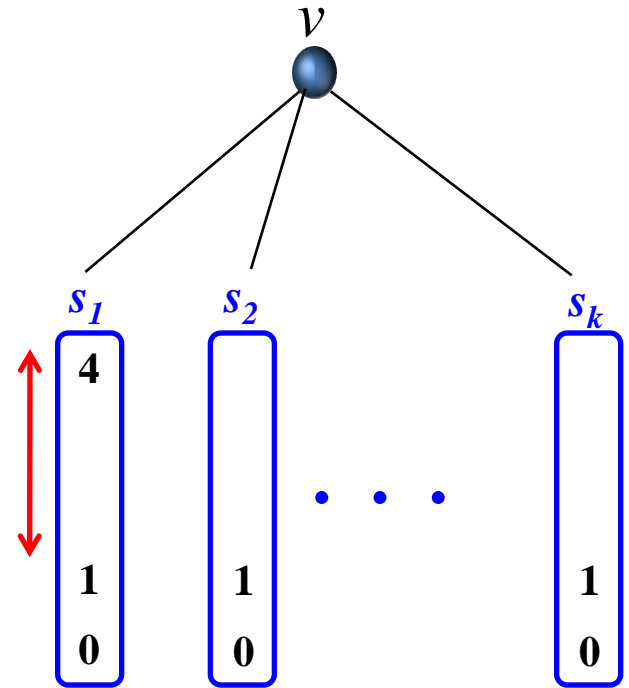
Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself



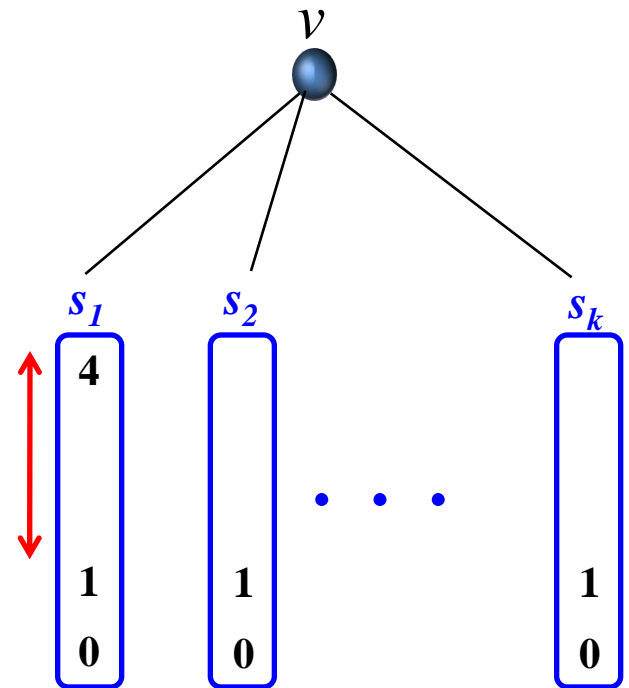
Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself
 - $k(v) = \#v$'s children



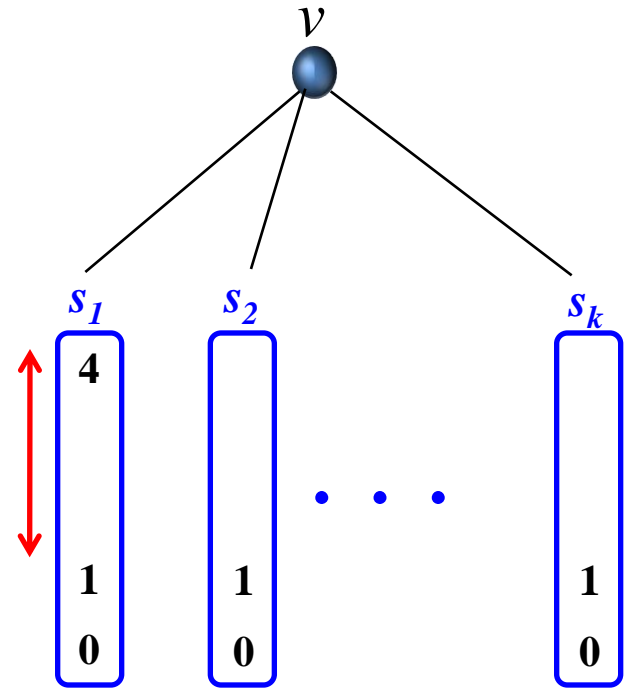
Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself
 - $k(v) = \#v$'s children
 - $q(v) = |S_2| + \dots + |S_k|$



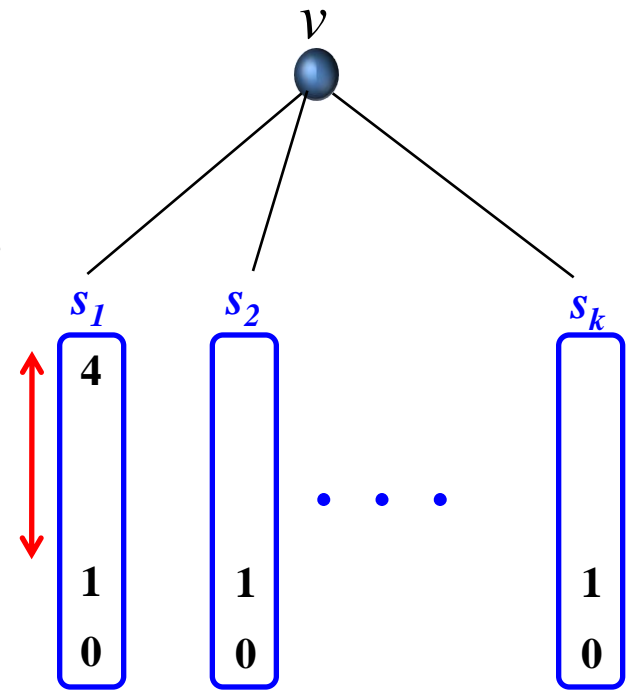
Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself
 - $k(v) = \#v$'s children
 - $q(v) = |S_2| + \dots + |S_k|$
 - $t(v) =$ largest value that appears in S_v but not in S_1



Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself
 - $k(v) = \#v$'s children
 - $q(v) = |S_2| + \dots + |S_k|$
 - $t(v) =$ largest value that appears in S_v but not in S_1
- an extension can be computed in $O(k(v) + q(v) + t(v))$



Running Time

- $|S_1| + |S_2| + \dots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at v itself

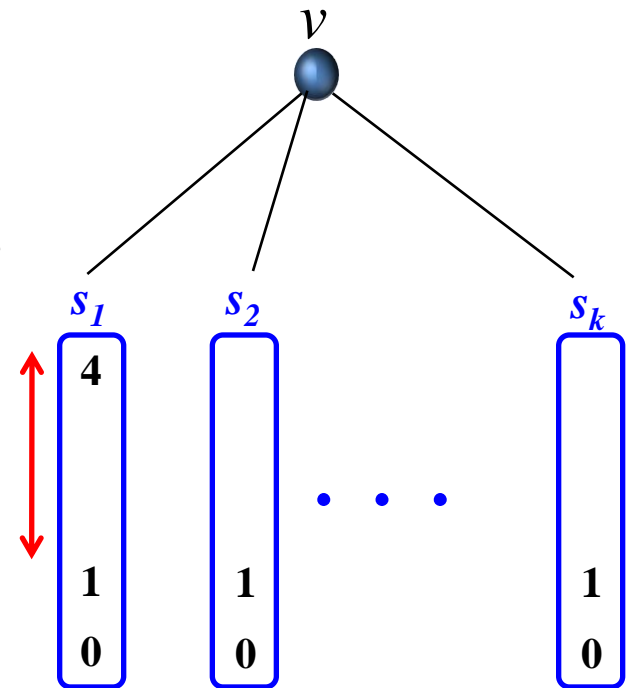
- $k(v) = \#v$'s children

- $q(v) = |S_2| + \dots + |S_k|$

- $t(v) =$ largest value that appears in S_v but not in S_1

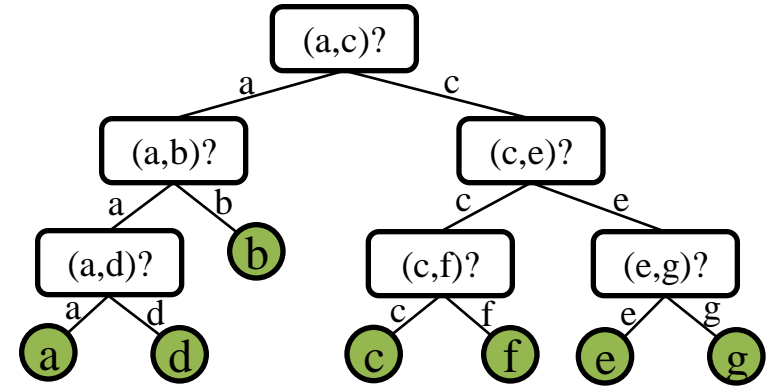
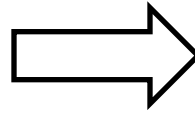
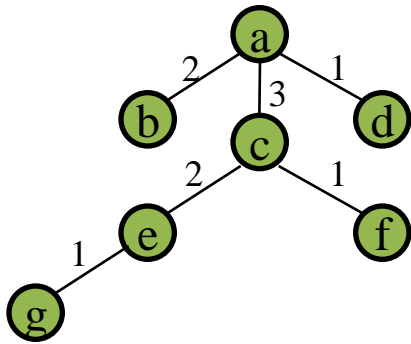
- an extension can be computed in $O(k(v) + q(v) + t(v))$

- $\sum_v k(v) + q(v) + t(v) = O(n)$

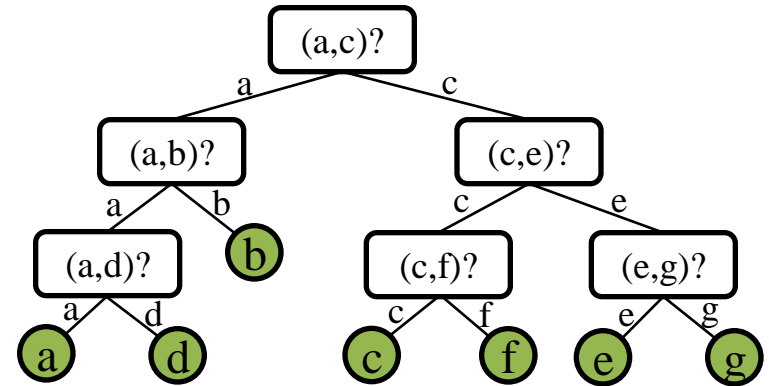
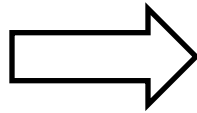
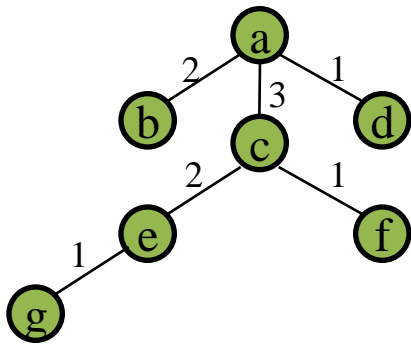


From Strategy Function to Decision Tree in $O(n)$ Time

From Strategy Function to Decision Tree in $O(n)$ Time



From Strategy Function to Decision Tree in $O(n)$ Time



● For all edges e

● let s = visibility sequence at $bottom(e)$

● if s contains no values smaller than $f(e)$

● set $bottom(e)$ as the solution when the query on e returns $bottom(e)$

● else, let $v_1 < \dots < v_k < f(e)$ in s , let e_i be the edge v_i is assigned to

● set e_k as the solution when the query on e returns $bottom(e)$

● for every $1 \leq i < k$ set e_i as the solution when the query on e_{i+1} returns $top(e_{i+1})$

● set $top(e_1)$ as the solution when the query on e_1 returns $top(e_1)$

Thank you !!

