1-Tough cocomparability graphs are hamiltonian

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Abstract

We show that every 1-tough cocomparability graph has a Hamilton cycle. This settles a conjecture of Chvátal for the case of cocomparability graphs. Our approach is based on exploiting the close relationship of the problem to the scattering number and the path cover number.

1. Introduction

The terminology we use is fairly standard except as indicated. For any undefined terms see e.g. [5]. We use $V(G)$, $E(G)$ and $c(G)$ to, respectively, denote the vertex set, the edge set and the number of components of a graph $G$. We always denote the number of vertices of a graph by $n$. Throughout the paper we assume $n \geq 3$. We consider only simple graphs, i.e., graphs without multiple edges or loops. A graph $H = (V(H), E(H))$ is a subgraph of the graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of the graph $G$ is a spanning subgraph of $G$ if $V(H) = V(G)$. For any $S \subseteq V(G)$, we write $G - S$ for the subgraph induced by $V(G) - S$ from $G$.

A partial order will be denoted by $P = (V, <_P)$, where $V$ is the finite ground set of elements or vertices and $<_P$ is an irreflexive, antisymmetric and transitive binary relation on $V$. Two elements $a, b \in V$ are comparable in $P$ if $a <_P b$ or $b <_P a$, otherwise they are said to be incomparable. The graph $G(P) = (V, E)$, whose edge set is exactly the set of comparable pairs in the partial order $P$ on $V$, is called the comparability graph of $P$. The complementary graph, whose edges are the incomparable...
pairs in $P$, is called the \textit{cocomparability graph} of $P$ and will be denoted by $G^c(P)$. Alternatively, a graph $G$ is a cocomparability graph if its complementary graph $G^c$ has a transitive orientation, corresponding to the comparable pairs of a partial order which is denoted by $P_G$. We note that a partial order uniquely determines its comparability graph and its cocomparability graph, but the reverse is not true, i.e., a cocomparability graph $G$ could have as many partial orders $P_G$, as is the number of the transitive orientations of $G^c$. We note that cocomparability graphs represent a relatively large class of perfect graphs, as this class properly contains both interval graphs and permutation graphs.

The \textit{toughness} of a graph $G$, denoted by $t(G)$, was defined by Chvátal [6]: For the complete graph $K_n$ we have $t(K_n) = \infty$; if $G$ is not complete, then $t(G) = \min\{|S|/c(G - S) : S \subseteq V(G) \text{ and } c(G - S) > 1\}$. A graph $G$ is said to be $t$-\textit{tough} if $t(G) \geq t$, i.e., $|S| \geq t \cdot c(G - S)$ for any cutset $S$. The toughness of a graph is closely related to the existence of Hamilton cycles in the graph. It is easy to see that if a graph has a Hamilton cycle then it must be 1-tough. In fact, $t(G) \geq 1$ has often been assumed as part of sufficient conditions for $G$ to be hamiltonian (cf. [3,4,11,13]). Chvátal conjectured in [6] that there is a positive constant $t_0$ such that every $t_0$-tough graph is hamiltonian. This conjecture remains open for general graphs, but it is shown in [9] that $t_0$ cannot be smaller than 2. In this paper we settle the conjecture for the class of cocomparability graphs, by showing that every 1-tough cocomparability graph is hamiltonian. A graph class where being 1-tough is also sufficient for a graph to be hamiltonian is called \textit{cycle-tough} [17]. The cycle-toughness of a hereditary graph class very well supports the design of a polynomial-time algorithm for deciding whether a given graph $G$ in the class is hamiltonian. This is because nonhamiltonicity has a good certificate, i.e., if $G$ is nonhamiltonian then it must have a cutset $S$ for which $G - S$ has more than $|S|$ components. This fact is used in polynomial time algorithms for deciding whether an interval graph is hamiltonian [15,20]. There are several recent papers proving Chvátal’s conjecture for different graph classes and $t_0$ values: $t(G) \geq 3/2$ is sufficient for a split graph to be hamiltonian [17]; $t(G) > 1$ is sufficient for circular-arc graphs [22]; and $t(G) \geq 1$ is sufficient for $P_4$-extendible graphs [12].

The problem of computing the toughness of a graph is an interesting problem on its own. Bauer et al. [2] have shown, however, that deciding whether $t(G) \geq t$ is coNP-complete for any fixed rational $t > 0$, i.e., it is NP-complete to decide whether there is a certificate $S \subseteq V(G)$ with $t \cdot c(G - S) > |S|$. This result was recently further sharpened in two directions: In [17] it was shown that deciding whether $t(G) \geq 1$ (or $t(G) = 1$) remains coNP-complete even if $G$ is bipartite; and in [1] it was shown that deciding whether $t(G) \geq t$ remains coNP-complete even if $\delta(G) \geq (t/(t+1) - \epsilon)n$, where $\epsilon > 0$ is an arbitrary constant and $\delta(G)$ denotes the minimum degree in $G$. Here we show that if $G$ is a cocomparability graph, then it can be determined in polynomial time whether $t(G) \geq 1$.

The \textit{scattering number} of a graph $G$, denoted by $sc(G)$, was defined by Jung [14] as follows: $sc(G) = \max\{c(G - S) - |S| : S \subseteq V(G) \text{ and } c(G - S) \neq 1\}$. Note that
for the complete graph \( K_n \) this yields \( \text{sc}(K_n) = -n \). A cutset \( S \) of a graph \( G \) fulfilling \( \text{sc}(G) = c(G - S) - |S| \) is said to be a scattering set. It follows from the definitions that \( t(G) \geq 1 \) if and only if \( \text{sc}(G) \leq 0 \) for any graph. The scattering number of a graph is closely related to the existence of Hamilton cycles and Hamilton paths. If a graph \( G \) has a Hamilton cycle then \( \text{sc}(G) \leq 0 \) must hold; and if a graph \( G \) has a Hamilton path then \( \text{sc}(G) \leq 1 \) must hold (see e.g. [5]). It is mentioned in [16] that deciding whether \( \text{sc}(G) \leq 0 \) (or \( \text{sc}(G) = 0 \)) is NP-complete for bipartite graphs. Polynomial time algorithms for computing the toughness and the scattering number of interval, circular-arc, permutation, trapezoid and circular permutation graphs, as well as cocomparability graphs of bounded dimension, are given in [16].

The path cover number of a graph \( G \), denoted by \( \pi(G) \), is the smallest number of disjoint paths covering the vertex set of \( G \). This parameter has attracted considerable attention recently [12,19]. Note, however, that it is closely related to problems of hamiltonicity, since for every graph \( \pi(G) - 1 = h_p(G) \), where \( h_p(G) \) is the well-known Hamilton path completion number, i.e., the minimum number of additional edges which would complete \( G \) into a graph with a Hamilton path. An \( O(n^2) \) algorithm computing \( h_p(G) \) was given in [7] for cocomparability graphs.

2. Preliminary results

Our results are based on exploiting the close relationship between the problems of finding a Hamilton path or cycle in a cocomparability graph and the bump number of a partial order corresponding to a transitive orientation of the complementary graph. Habib et al. [10] and independently Schäffer and Simons [21] have found polynomial algorithms for computing the bump number of a general partial order. Here we use the notation and the duality theory developed in [10].

A linear order is a partial order with no incomparable elements. A linear extension of a partial order \( P = (V, <_P) \) is a linear order \( L = (V, <_L) \) on the same ground set that extends \( P \), i.e., \( a <_P b \Rightarrow a <_L b \) for all \( a, b \in V \). We will write linear orders as \( L = v_1, v_2, \ldots, v_n \), where the sequence defines the order relation, i.e., \( v_1 <_L v_2 <_L \cdots <_L v_n \). Two consecutive elements \( v_i, v_{i+1} \) of a linear extension \( L \) are separated by a bump if \( v_i <_P v_{i+1} \). The total number of bumps in \( L \) is denoted by \( b(L, P) \). The bump number \( b(P) \) of a partial order \( P \) is the minimum number of bumps in some linear extension, i.e.,

\[
b(P) = \min\{b(L, P) : L \text{ is a linear extension of } P\}.
\]

A set of incomparable elements in \( P \) is called an antichain. \( P^* = (V, <_{P^*}) \) is a generalized weak order from the class \( \mathcal{P}_{r,s} \) if it is the parallel composition (disjoint union) of an antichain \( A_0 \) on \( s \) elements with the series composition (ordinal sum) of \( r \) antichains \( A_1, A_2, \ldots, A_r \). In other words, \( V \) partitions into \( V = V_0 \cup V_1 \cup \cdots \cup V_r \), \( |V_0| = s \) and \( u <_{P^*} v \) if and only if \( u \in V_i \) and \( v \in V_j \) for some \( 1 \leq i < j \leq r \). It can be easily seen that if \( P^* \in \mathcal{P}_{r,s} \) then \( b(P^*) = \max\{0, r - 1 - s\} \).
The HMS algorithm given in [10] is of the primal-dual type. Primal solutions are linear extensions of the given order \( P \), and the dual solutions correspond to generalized weak orders contained in \( P \). Containment here is defined with respect to the order relations. \( P \) and \( P^* \) are partial orders on the same ground set \( V \), and \( P^* \) is contained in \( P \) or equivalently, \( P \) extends \( P^* \), if \( a <_P b \) implies \( a <_P b \) for all \( a, b \in V \). It is clear that for any linear extension \( L \) of \( P \) and generalized weak order \( P^* \) contained in \( P \) we have

\[
 b(P^*) \leq b(P) \leq b(L, P). \tag{1}
\]

The following theorem summarizes the main results from [10], in a form suitable for our use:

**Theorem 1.** For any partial order \( P \)

1. \( \max\{b(P^*) : P^* \text{ a generalized weak order contained in } P\} = \min\{b(L, P) : L \text{ a linear extension of } P\} \);

2. there is an \( O(n^2) \) algorithm to find the optimal \( L \) and \( P^* \) for which equality holds in (1).

The HMS algorithm constructs \( L \) and \( P^* \) such that \( b(P^*) = b(L, P) \), thus implying equality in (1). The generalized weak order produced always satisfies \( s < r \), therefore \( b(P^*) = r - 1 - s \).

If \( G \) is a cocomparability graph and \( P_G \) is a corresponding partial order on \( V(G) \), then \( b(P_G) \) does not depend on which one of the transitive orientations was chosen for \( G^c \) to get \( P_G \), i.e., \( b(P_G) \) is a (co)comparability invariant, so it can be viewed as a parameter of \( G \). In fact, we proved the following much stronger result in [7].

**Theorem 2.** If \( G \) is a cocomparability graph, then \( b(P_G) + 1 = \pi(G) \).

We note that Theorem 2 implies that a cocomparability graph \( G \) has a Hamilton path if and only if \( b(P_G) = 0 \). Thus, the \( O(n^2) \) algorithm solving the bump number problem automatically yielded a \( O(n^2) \) algorithm for finding a Hamilton path in a cocomparability graph, if there is one, or proving that there does not exist one. Interestingly, it turned out to be much more difficult to apply the method to the Hamilton cycle problem in cocomparability graphs. Nevertheless, the same approach works, and a long proof was given for the following theorem in [8].

**Theorem 3.** A cocomparability graph \( G \) has a Hamilton cycle if and only if \( G \) has a Hamilton path and for all \( v \in V(G) \) the graph \( G - v \) also has a Hamilton path.

The main purpose of this paper is to show how to use the above-mentioned results for proving that every 1-tough cocomparability graph has a Hamilton cycle.
3. The scattering number

We first give several technical lemmas concerning the monotonicity of the scattering number in graphs in general. We recall that the definition of the scattering number implies that $sc(G) \leq sc(H)$ for any spanning subgraph $H$ of a graph $G$.

**Lemma 4.** $sc(G) \geq sc(G - V') - |V'|$ holds for every subset $V' \subseteq V(G)$ in any graph $G$.

**Proof.** If $G - V'$ is complete, then the inequality obviously holds. Let $V'$ be a subset of $V(G)$ such that $G - V'$ is not complete and let $S'$ be a scattering set of $G - V'$. Consequently, $V' \cup S'$ is a cutset of $G$. Hence, $sc(G) \geq c(G - (V' \cup S')) - |V' \cup S'| = c((G - V') - S') - |S'| - |V'| = sc(G - V') - |V'|$. $\square$

We will mainly use the special case $V' = \{v\}$.

**Lemma 5.** $sc(G) \geq sc(G - v) - 1$ for every vertex $v \in V(G)$ in any graph $G$.

The similar statement for an upper bound on $sc(G)$ in terms of $sc(G - v)$ is significantly weaker.

**Lemma 6.** Let $G$ be a connected graph. Then there is a vertex $v \in V(G)$ such that $sc(G - v) - 1 \geq sc(G)$.

**Proof.** The lemma is clearly true for complete graphs. So suppose the graph $G$ is not complete and let $S$ be a scattering set of $G$. Since $G$ is connected $S$ is nonempty. We choose a vertex $v \in S$. Then $S - \{v\}$ is a cutset of $G - v$ and $sc(G - v) \geq c((G - v) - (S - \{v\})) - |S - \{v\}| = c(G - S) - |S| + 1 = sc(G) + 1$. $\square$

Lemmas 5 and 6 immediately imply the following proposition, which is interesting on its own.

**Proposition 7.** For every connected graph $G$

$$sc(G) = \max_{v \in V(G)} sc(G - v) - 1.$$
vertices from at most \( r + 1 \) components of \( G - S \). Hence, there must be at least \( \text{sc}(G) \) paths in any cover of \( V(G) \) by disjoint paths, i.e., \( \tau(G) \geq \text{sc}(G) \).

Let \( G \) be a cocomparability graph with \( \tau(G) \geq 2 \). By Theorem 2 we have \( \tau(G) = b(P_G) + 1 \). By Theorem 1, there is a generalized weak order \( P^* \in \mathcal{P}_{s,r} \) with \( P^* \) contained in \( P_G \) and \( b(P^*) = b(P_G) = r - s - 1 = \pi(G) - 1 \). Note that \( \pi(G) \geq 2 \) implies \( r - s \geq 2 \) and so \( r \geq 2 \).

It can be easily seen from the definition of generalized weak orders that the vertex set \( V \) of \( G^*(P^*) \) partitions into \( V = V_0 \cup V_1 \cup \cdots \cup V_r \) with the following properties:

- \( V_0 \) induces a clique in \( G^*(P^*) \) and \( |V_0| = s \);
- each \( V_i \) induces a clique in \( G^*(P^*) \);
- there exists no edge \((u, v)\) in \( G^*(P^*) \) with \( u \in V_i, v \in V_j, 1 \leq i < j \leq r \);
- \((u, v)\) is an edge in \( G^*(P^*) \) for every \( u \in V_0 \) and \( v \in V_j, 1 \leq j \leq r \).

Consequently, \( V_0 \) is a cutset of \( G^*(P^*) \), \( |V_0| = s \) and \( c(G^*(P^*) - V_0) = r \), since \( G^*(P^*) - V_0 \) is the disjoint union of the \( r \) cliques \( V_1, V_2, \ldots, V_r \). Hence, \( \text{sc}(G^*(P^*)) \geq r - s \). The cocomparability graph of \( P_G \) is a spanning subgraph of the cocomparability graph of \( P^* \), since \( P^* \) is contained in \( P_G \). Therefore, \( \text{sc}(G) \geq \text{sc}(G^*(P^*)) \geq r - s = \pi(G) \). \( \square \)

Note that a corresponding theorem was shown in [12] for \( P_4 \)-extendible graphs.

4. The toughness of cocomparability graphs

First we present the following strengthening of Theorem 3.

**Theorem 9.** A cocomparability graph \( G \) has a Hamilton cycle if and only if \( G - v \) has a Hamilton path for all \( v \in V(G) \).

**Proof.** By Theorem 3 we only have to show that \( G \) has a Hamilton path if for all \( v \in V(G) \) \( G - v \) has a Hamilton path.

Contrary to the statement, assume that \( G \) does not have a Hamilton path, i.e., \( \tau(G) \geq 2 \). By Theorem 8 we have \( \text{sc}(G) \geq 2 \). \( G \) is connected, since otherwise there would be a \( v \in V(G) \) such that \( G - v \) had no Hamilton path. \( G \) is not complete, since \( G \) has no Hamilton path. Now we can apply Lemma 6, which guarantees the existence of a vertex \( v \) of \( G \) such that \( \text{sc}(G - v) - 1 \geq \text{sc}(G) = 2 \). Hence, \( \text{sc}(G - v) \geq 3 \) and \( G - v \) has no Hamilton path, a contradiction. \( \square \)

Due to all the preparing lemmas and theorems, it is now easy to prove the main theorem of our paper.

**Theorem 10.** A cocomparability graph \( G \) has a Hamilton cycle if and only if \( t(G) \geq 1 \).

**Proof.** \( t(G) \geq 1 \) is a necessary condition for any graph \( G \) to have a Hamilton cycle [6].
For the other direction, let \( t(G) \geq 1 \). This implies \( sc(G) \leq 0 \). By Lemma 5, for every vertex \( v \in V(G) \) we have \( sc(G) \geq sc(G - v) - 1 \). Hence, \( sc(G - v) \leq 1 \) for every \( v \in V(G) \). From Theorem 8 we get that for every \( v \in V(G) \) the graph \( G - v \) has a Hamilton path. Finally, following Theorem 9, this implies that \( G \) has a Hamilton cycle. □

**Corollary 11.** A cocomparability graph \( G \) has a Hamilton cycle if and only if \( sc(G) \leq 0 \).

Theorem 10 also implies that for a cocomparability graph it can be decided in polynomial time whether it is 1-tough.

**Corollary 12.** It can be decided in \( O(n^3) \) time for a cocomparability graph \( G \) whether \( t(G) \geq 1 \).

**Proof.** By Theorems 9 and 10, \( t(G) \geq 1 \) if and only if \( G - v \) has a Hamilton path for every \( v \in V \). By Theorem 2 this, however, is equivalent to \( b(P_{G-v}) = b(P_G - v) = 0 \) for every \( v \in V \). The bump number algorithm [10] requires \( O(n^2) \) time to find the bump number for a partial order on \( n \) elements. □

We also note that if the bump number algorithm finds a \( u \in V \) such that \( b(P_G - u) > 0 \), then it exhibits the dual solution too, i.e., the generalized weak order \( P^* \in P_{r,s} \) contained in \( P_G - u \) for which \( b(P_G - u) = b(P^*) = r - s - 1 \). If \( V_0 \) is the set of \( s \) isolated elements in \( P^* \), then \( V_0 \) is a cutset in \( G^c(P^*) \) and \( G^c(P^*) - V_0 \) has \( r > s + 1 \) components. Consider now \( S = V_0 \cup \{ u \} \). \( S \) is a cutset in \( G \) and \( G - S \) has \( r > s + 1 = |S| \) components, so \( S \) is a certificate that \( G \) is not 1-tough.

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**References**