A Characterization of Ptolemaic Graphs*

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ABSTRACT

A connected graph $G$ is *ptolemaic* provided that for each four vertices $u_i$, $1 \leq i \leq 4$, of $G$, the six distances $d_{ij} = d_G(u_i, u_j)$, $i \neq j$ satisfy the inequality $d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23}$ (shown by Ptolemy to hold in Euclidean spaces). Ptolemaic graphs were first investigated by Chartrand and Kay, who showed that weakly geodetic ptolemaic graphs are precisely Husimi trees (in particular, trees are ptolemaic). In the present paper several characterizations of ptolemaic graphs are given. It is shown, for example, that a connected graph $G$ is ptolemaic if and only if for each nondisjoint cliques $P, Q$ of $G$, their intersection is a cutset of $G$ which separates $P-Q$ and $Q-P$. An operation is exhibited which generates all finite ptolemaic graphs from complete graphs.

The graphs considered in this paper are undirected, without loops or multiple edges. The basic terminology is that of [1]. Unless stated otherwise, the graphs are not assumed to be finite.

A metric space $(X, d)$ is *ptolemaic* if for each four elements $u_i$, $1 \leq i \leq 4$, of $X$, the six distances $d_{ij} = d(u_i, u_j)$, $i \neq j$, satisfy the *ptolemaic inequality*: $d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23}$ (shown by Ptolemy to hold in Euclidean spaces).

The notion of a ptolemaic space was introduced by Blumenthal [4] (see also [5], p. 78). In [6] Chartrand and Kay defined ptolemaic graphs: a graph $G$ is *ptolemaic* if $G$ is connected and if the metric space $(V(G), d_G)$ associated with $G$ (see 1.2) is ptolemaic.

It is well known that the only graphs isometrically embeddable in a Euclidean space are open (i.e., acyclic) paths and finite complete graphs—a narrow class indeed. The ptolemaic inequality may be regarded, therefore, as a naturally weaker condition which nevertheless preserves certain remarkable properties of the distance. A restricted class of *weakly geodetic* ptolemaic graphs was shown in [6] to be exactly the class of *Husimi trees* (see also [8]). In particular, trees are ptolemaic. In the present paper, a detailed description of ptolemaic graphs is given. Section 1 is preliminary.

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Section 2 contains the basic characterization (Theorem 2.1) which states that a graph is ptolemaic iff it is distance hereditary and triangulated. Among its corollaries, the most important is Theorem 2.4. In Sec. 3, finite ptolemaic graphs are characterized as those graphs which can be obtained from complete graphs by identifications of certain sets of vertices. Some results of this paper were announced in [10], where the reader may also find certain characterizations not restated here. A few results describing the relationship between ptolemaic and other classes of graphs are given in [8].

1. PRELIMINARIES

1.1 Throughout the paper, paths are denoted by lower-case Greek letters. A path $\alpha = (a_0, a_1, \ldots, a_n)$ of a graph $G$ is a cycle if $a_0 = a_n$ and $n > 0$; $\alpha$ is an open path otherwise. The integer $n$ is the length of $\alpha$, denoted by $l(\alpha)$. A cycle of length $n = 3$ is also called a triangle. By forgetting about the orientation of a path $\alpha$, we identify $\alpha$ with that subgraph of $G$ having the same vertices and edges as $\alpha$. Let $F$ be an arbitrary subgraph of $G$, and let $u, v$ be distinct vertices of $F$ adjacent in $G$. The edge $uv$ of $G$ will be called a triangular chord of $F$ (in $G$) provided that there exists a vertex $x$ of $F$ distinct from $u$ and $v$ such that both edges $xu$ and $xv$ are in $F$. $F$ is said to be a strong subgraph of $G$ if it contains all of its triangular chords. The subgraph of $G$ induced by the vertices of a subgraph $F$ is denoted by $\langle F \rangle$. $F$ is an induced subgraph of $G$ if $F = \langle F \rangle$. Clearly, an induced subgraph of $G$ is strong in $G$.

We define an induced (resp. strong) path of $G$ to be a path which is an induced (resp. strong) subgraph of $G$. For example, a path different from a triangle is strong iff it has no triangular chords. Let $\alpha$ be a cycle of $G$; an edge contained in $\langle \alpha \rangle$ but not in $\alpha$ is a diagonal of $\alpha$. $G$ is triangulated [2] if each cycle of $G$ of length $\geq 4$ has a diagonal. (Triangulated graphs are called rigid circuit graphs by some authors.)

1.2. Let $G$ be a connected graph. The distance $d_G(u, v)$ between two vertices of $u, v$ of $G$ is the length of any shortest $u - v$ path of $G$. $(V(G), d_G)$ is the metric space associated with $G$. A $u - v$ path $\alpha$ of $G$ such that $l(\alpha) = d_G(u, v)$ is called a $u - v$ geodesic. Clearly, any geodesic is open and induced, hence strong, but the converse is generally false. A detour is a path which is not a geodesic. A connected subgraph $F$ of $G$ such that $d_F(u, v) = d_G(u, v)$ for each $u, v \in F$, is called a metric subgraph. In other words, $F$ is a metric subgraph of $G$ iff $F$ is connected and $(V(F), d_F)$ is a metric subspace of $(V(G), d_G)$. Clearly, a metric subgraph is induced. $G$ is called distance hereditary if the converse is true as well, i.e., if each connected induced subgraph of $G$ is metric. Equivalently, $G$ is distance hereditary provided that every open induced path of $G$ is geodesic. The following characterization of distance-hereditary graphs ([9], Theorem 1) will be useful in the sequel:
Theorem 1.3. \( G \) is distance hereditary if and only if each cycle of \( G \) of length \( \geq 5 \) has at least two diagonals, and each 5-cycle of \( G \) has two diagonals which cross one another.

1.4. Let \( G \) be a connected graph. A set of vertices of \( G \) is a cutset of \( G \) if the graph \( G-S \) is disconnected. \( S \) is a minimal cutset of \( G \) if, in addition, no proper subset of \( S \) is a cutset of \( G \). Let \( u,v \) be vertices of \( G \). We will say that \( S \) separates \( u \) from \( v \), or that \( S \) is a \( u-v \) cutset, if \( u \) and \( v \) lie in distinct components of \( G-S \). If, moreover, no proper subset of \( S \) has this property, then \( S \) is a relatively minimal \( u-v \) cutset. Let \( U,V \) be sets of vertices of \( G \). \( S \) is said to separate \( U \) and \( V \) provided that \( S \) separates each element of \( U \) from every element of \( V \).

Our main Theorem relies on the following characterizations of triangulated and distance-hereditary graphs in terms of relatively minimal cutsets. The first one is due to Dirac ([7], Theorem 1; see also [3]).

Theorem 1.5. (Dirac). A graph \( G \) is triangulated if and only if every relatively minimal cutset of \( G \) induces a complete graph.

Let \( u,v \) be vertices of a connected graph \( G \). Put \( n = d_G(u,v) \). For each \( k = 1, \ldots, n-1 \), let \( L_k(u,v) \) denote the set \( \{ x : d_G(u,x) = k, d_G(v,x) = n-k \} \). The sets \( L_k(u,v) \) will be called \( u-v \) levels.

Theorem 1.6. (A characterization of distance-hereditary graphs) Let \( G \) be a connected graph. The following propositions are equivalent:

(i) \( G \) is distance hereditary;

(ii) for each two vertices \( u,v \) of \( G \), every \( u-v \) level is a \( u-v \) cutset;

(iii) for each two vertices \( u,v \) of \( G \), every relatively minimal \( u-v \) cutset is a \( u-v \) level.

Proof: (i) \( \Rightarrow \) (ii). Let \( u,v \) be vertices of \( G \), and let \( S \) be a set of vertices of \( G \) which is not a \( u-v \) cutset. Let \( \alpha \) be any shortest \( u-v \) path in \( G-S \). Clearly, \( \alpha \) is an induced path of \( G \). Since \( G \) is distance hereditary, \( \alpha \) is a \( u-v \) geodesic of \( G \). It follows that \( S \) is not a \( u-v \) level.

3(i) \( \Rightarrow \) (iii). Let \( u,v \) be vertices of \( G \) and let \( S \) be a relatively minimal \( u-v \) cutset of \( G \). Thus there is a \( u-v \) path of \( G \) which has only one vertex in common with \( S \). Let \( \alpha = (u = \alpha_0, a_1, \ldots, a_m = v) \) be such a path of minimum length. Clearly, \( \alpha \) is an induced path of \( G \). Since \( G \) is distance hereditary, \( \alpha \) is a geodesic. Let \( a_k \) be the unique vertex of \( \alpha \) in \( S \). Consider the set \( L = L_1(a_{k-1}, a_{k+1}) \). Clearly, \( L \subseteq S \) and \( L \subseteq L_k(u,v) \). On the other hand, by (ii), \( L \) is an \( a_{k-1} - a_{k+1} \) cutset, hence also a \( u-v \) cutset. Thus, by minimality of \( S \), \( L = S \). Finally, since no \( u-v \) cutset can be properly contained in a \( u-v \) level, we have \( S = L_k(u,v) \).

(ii) \( \Rightarrow \) (i), (iii) \( \Rightarrow \) (i). Assume that \( G \) is not distance hereditary. Let \( \alpha \) be an induced open detour of minimal length in \( G \). Let \( u,v \) be end points of \( \alpha \). Since \( \alpha \) is induced, \( d_G(u,v) \geq 2 \). Hence there exists a \( u-v \) level and a \( u-v \)
cutset of $G$. On the other hand, since each proper subpath of $\alpha$ is a geodesic, it follows that $\alpha \cap L = \emptyset$ for each $u-v$ level $L$. This violates (ii). To obtain a contradiction with (iii), note that by Haussdorff's minimality principle, each $u-v$ cutset contains a relatively minimal $u-v$ cutset.

2. CHARACTERIZATIONS

**Main Theorem 2.1.** Let $G$ be a connected graph. The following propositions are equivalent:

(i) $G$ is ptolemaic;
(ii) every strong open path of $G$ is a geodesic;
(iii) $G$ is distance hereditary and triangulated;
(iv) for each two vertices $u, v$ of $G$, every relatively minimal $u-v$ cutset is a $u-v$ level and induces a complete subgraph.

**Proof.** (i) $\Rightarrow$ (ii) Assume the contrary. Let $\alpha = (a_0, a_1, \ldots, a_k)$ be a strong open detour of minimal length in $G$. Put $k = d_G(a_0, a_k)$. By assumption on $a$, each proper subpath of $\alpha$ is a geodesic. Thus $d_G(a_1, a_n) = d_G(a_0, a_{n-1}) = n - 1$. It follows that $n = k + 1$ or $n = k + 2$. It is easy to verify now that the vertices $a_0, a_{n-1}, a_n, a_1$ (in that order) violate the ptolemaic inequality. (Note that $n \geq 3$.) This contradiction completes the proof.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Follows from Theorems 1.5 and 1.6.

(iv) $\Rightarrow$ (i). Let $G$ be a connected graph satisfying (iv) and let $W = \{u_1, \ldots, u_4\}$ be a set of four (not necessarily distinct) vertices of $G$. Write $d_{ij} = d_G(u_i, u_j)$ and let $d(W) = \sum d_{ij}$ $(1 \leq i < j \leq 4)$. To say that $W$ satisfies the ptolemaic inequality means that all of the following inequalities hold:

\[
d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23} \quad (1)
\]
\[
d_{13}d_{24} \leq d_{12}d_{34} + d_{14}d_{23} \quad (2)
\]
\[
d_{14}d_{23} \leq d_{12}d_{34} + d_{13}d_{24} \quad (3)
\]

To prove (i) it is convenient (and obviously sufficient) to show that any set of four vertices of $G$ satisfies the ptolemaic inequality. The proof will proceed by induction on $d(W)$. Consider a fixed set $W = \{u_1, \ldots, u_4\}$, and suppose that each set $W'$ of four vertices of $G$ such that $d(W') < d(W)$ satisfies the ptolemaic inequality. Observe first that if $|W| = 4$, or if two vertices of $W$ coincide, then $W$ trivially satisfies the ptolemaic inequality. Assume henceforth that neither possibility takes place.

Let $S$ be a set of vertices of $G$ which is minimal with respect to the property that $W-S$ is not entirely contained in one same component of $G-S$. (By assumption on $W$, such a set $S$ exists, even if $G$ if infinite.) If $u$ and $v$ are elements of $W-S$ separated by $S$, then $S$ is a relatively minimal $u-v$ cutset; hence, by (iv), $S$ is a $u-v$ level which induces a complete subgraph of $G$. 
Suppose now that $S$ contains precisely two vertices of $W$, say $u_1, u_2$. Thus $u_1$ and $u_2$ are adjacent, $d_{1i} = d_{2i}$ ($i = 3, 4$) and $d_{3i} + d_{4i} = d_{34}$ ($i = 1, 2$). Consequently, an easy verification shows that $W$ satisfies the ptolemaic inequality. Hence, it can be assumed that $S$ contains at most one vertex of $W$. Let $v$ denote this vertex if it exists; otherwise, let $v$ denote an arbitrarily chosen vertex of $S$. Write $d_i = d_G(u_i, v)$ for each $u_i \in W$. Note that if $u_i, u_j$ are elements of $W$ not both contained in the same component of $G - S$, then

$$d_{ij} = d_i + d_j.$$  \hspace{1cm} (4)

To complete the proof, we shall consider two cases. The induction hypothesis will be used only in the second case.

**Case I.** A component $C$ of $G - S$ contains precisely two elements of $W$. Since the labeling is inessential, we may assume that $u_1, u_2$ are in $C$. Thus, by (4),

$$d_{ij} = d_i + d_j \hspace{1cm} (i = 1, 2; j = 3, 4)$$  \hspace{1cm} (5)

By substituting (5) and regrouping the terms, the inequalities (1)–(3) may now be rewritten as follows:

$$d_{12}d_{34} \leq (d_1 + d_2)(d_3 + d_4) + 2(d_1d_2 + d_3d_4)$$  \hspace{1cm} (1')

$$(d_1 - d_2)(d_4 - d_3) \leq d_{12}d_{34}$$  \hspace{1cm} (2')

$$(d_1 - d_2)(d_3 - d_4) \leq d_{12}d_{34}$$  \hspace{1cm} (3')

Clearly, each of the inequalities (1')–(3') holds in virtue of the triangle inequality. Hence $W$ satisfies the ptolemaic inequality.

**Case II.** A component $C$ of $G - S$ contains precisely one vertex. Without loss of generality, we may assume that, say, $u_1$ is in $C$. By (4),

$$d_{1j} = d_1 + d_j \hspace{1cm} (j = 2, 3, 4).$$  \hspace{1cm} (6)

Clearly, $d((v, u_2, u_3, u_4)) \leq d(W)$. Thus, by induction hypothesis, the set $\{v, u_2, u_3, u_4\}$ satisfies the ptolemaic inequality. In particular, by (1),

$$d_2d_{34} \leq d_3d_{24} + d_4d_{23}.$$  \hspace{1cm} (7)

On the other hand, by the triangle inequality,

$$d_{34} \leq d_{24} + d_{23}.$$  \hspace{1cm} (8)
After multiplying both sides of (8) by $d_i$ and adding the resulting inequality to (7), we have

$$(d_i + d_2)d_{34} \leq (d_1 + d_3)d_{24} + (d_1 + d_4)d_{23}.$$ 

By (6), we obtain (1). The proofs of (2) and (3) are completely analogous. This proves that $W$ satisfies the ptolemaic inequality.

The proof of the theorem is now complete. \qed

The condition (ii) of the main Theorem yields the following:

**Corollary 2.2.** A connected graph $G$ is ptolemaic if and only if each strong connected subgraph of $G$ is metric.

**Corollary 2.3.** Each connected strong subgraph of a ptolemaic graph is ptolemaic. In particular, each connected induced subgraph of a ptolemaic graph is ptolemaic.

For the sake of clarity, strong characterizations were excluded from the main Theorem. The following one will be useful in the sequel.

**Theorem 2.4.** A connected graph $G$ is ptolemaic if and only if for each distinct nondisjoint cliques $P, Q$ of $G$, $P \cap Q$ separates $P - Q$ and $Q - P$.

**Proof.** Let $P, Q$ be distinct nondisjoint cliques of a ptolemaic graph $G$. Choose nonadjacent vertices $p \in P$ and $q \in Q$, and assume that $P \cap Q$ does not separate $p$ and $q$. Since $G$ is distance hereditary, $d_{G-(P \cap Q)}(p, q) = 2$. Hence, there exists $v$ in $G - (P \cap Q)$, say $v \notin P$, such that $v$ is adjacent to both $p$ and $q$. Since $v \notin P$, there exists $u \in P$ such that $uv \notin G$. In particular, $u \neq p$ and $u \neq q$. Now, the path $(u, p, v, q)$ is a strong detour, in contradiction with 2.1.(ii). This proves the "only if" part of the theorem. Assume conversely that $G$ is a connected graph satisfying the condition of the theorem. Let $u, v$ be nonadjacent vertices of $G$, and let $1 \leq k \leq d_G(u, v) - 1$. Let $P, Q$ be cliques containing respectively the $k$th and the $(k + 1)$th edges of some $u-v$ geodesic. Clearly, $P \cap Q \subseteq L_k(u, v)$. Since $P \cap Q$ separates $P - Q$ and $Q - P$, it also separates $u$ and $v$. Hence $P \cap Q = L_k(u, v)$. It follows that every $u-v$ level of $G$ is a $u-v$ cutset which induces a complete subgraph. Applying Theorems 1.6 and 2.1, we conclude that $G$ is ptolemaic. \qed

The proof of the following theorem is left to the reader. Statements of other characterizations of ptolemaic graphs may be found in [10].

**Theorem 2.5.** A connected graph $G$ is ptolemaic if and only if for each $n \geq 3$, every $n$-cycle of $G$ has at least $\lfloor \frac{1}{2}(n - 3) \rfloor$ diagonals.
3. THE CLIQUE STRUCTURE OF PTOLEMAIC GRAPHS

The main result of this section is the recursive characterization of finite ptolemaic graphs presented in Theorem 3.8.

Definition 3.1. A graph $G$ is called semiptolemaic if, for every subgraph $H$ (see Fig. 1) of $G$, either $xb \in G$ or $ya \in G$.

It was shown in [8], Theorem 3.2, that $G$ is semiptolemaic if and only if the set of cliques of $G$ has the following property $\mathcal{F}$: for each finite collection $\mathcal{X}$ of pairwise nondisjoint cliques of $G$, and for each $Q \in \mathcal{X}$, there exists $P \in \mathcal{X}$ such that $Q \cap P = \bigcap \mathcal{X}$. (The wording is trivially different from that given in [8]).

Theorem 3.2. A connected graph $G$ is ptolemaic if and only if $G$ is semiptolemaic and triangulated.

Proof. To prove the "if" part, note that a 5-cycle of a graph which is semiptolemaic and triangulated must have two diagonals which cross one another. Thus, by Theorems 1.3 and 2.1, the graph is ptolemaic. Conversely, a ptolemaic graph is semiptolemaic (otherwise the vertices $x, b, a, y$ of $H$, Fig. 1, would violate the ptolemaic inequality) and, by 2.1.(iii), triangulated. □

Corollary 3.3. Let $\mathcal{X}$ be a finite collection of at least two distinct and pairwise nondisjoint cliques of a ptolemaic graph $G$. Then $\bigcap \mathcal{X}$ is a cutset of $G$.

Proof. The assertion follows from the fact that $G$ has the property $\mathcal{F}$ and from Theorem 2.4. □

Definition 3.4. Let $G$ be a graph. The minimal (with respect to inclusion) nonempty elements of the boolean algebra of subsets of $V(G)$ generated by the collection of cliques of $G$ will be called atoms of $G$. An atom which is called an intersection of a finite number $\geq 2$ of cliques will be called an
intersection atom. Clearly, if $G$ is finite, then the atoms partition $V(G)$. Vertices $u, v$ of $G$ are then in the same atom provided that either $u = v$ or $uv \in E$, and for each vertex $x$ of $G$ distinct from $u$ and $v$, $xu \in E$ iff $xv \in E$.

The following theorem is a direct consequence of Corollary 2.3.

**Theorem 3.5.** Let $S$ be an intersection atom of a ptolemaic graph $G$. Then $S$ is a cutset of $G$.

**Definition 3.6.** A nonempty subset of an atom of $G$ will be called a subatom of $G$. Let $G_1, G_2$ be disjoint graphs, and let $S_1, S_2$ be subatoms of $G_1, G_2$, respectively, having the same cardinality. Let $f$ be a one-to-one correspondence between the vertices of $S_1$ and the vertices of $S_2$. Consider the graph $G$ obtained from $G_1$ and $G_2$ by identifying $S_1$ with $S_2$ through $f$. Clearly, $G$ depends, up to an isomorphism, only on the choice of $S_1$ and $S_2$ and not on the choice of $f$. We will say that $G$—or any graph isomorphic to $G$—is obtained by a subatomic identification from $G_1$ and $G_2$.

**Theorem 3.7.** Let $G$ be a graph obtained by a subatomic identification from graphs $G_1$ and $G_2$. Then $G$ is ptolemaic if and only if both $G_1$ and $G_2$ are ptolemaic.

**Proof:** Consider $G_1$ and $G_2$ as subgraphs of $G$. By construction, each clique of $G$ is either a clique of $G_1$ or a clique of $G_2$. Moreover, $S = G_1 \cap G_2$ is a subatom of each $G_i$; hence $S$ is an intersection atom of $G$. It follows now easily from Theorem 2.4 that $G$ is ptolemaic if both $G_i$'s are ptolemaic. The converse follows from Corollary 2.3.

We will show now that a finite graph is ptolemaic if and only if it can be obtained by a series of subatomic identifications from complete graphs. More precisely, we have

**Theorem 3.8.** (Recursive characterization of ptolemaic graphs.) Let $\mathcal{P}$ denote the class of all finite ptolemaic graphs. Let $\mathcal{C}$ be the smallest class of graphs satisfying the following two conditions:

(i) $\mathcal{C}$ contains all finite graphs;

(ii) whenever $G_1, G_2 \in \mathcal{C}$, and $G$ is a graph obtained from $G_1$ and $G_2$ by a subatomic identification, then $G \in \mathcal{C}$.

Then $\mathcal{P} = \mathcal{C}$.

**Proof:** We first show by induction that $\mathcal{P} \subseteq \mathcal{C}$. Let $G \in \mathcal{P}$, and suppose that all the ptolemaic graphs having fewer vertices than $G$ are members of $\mathcal{C}$. Without loss of generality, we may assume that $G$ is not a complete graph. Thus, since $G$ is finite, it has an intersection atom $S$. By Theorem 3.5, $S$ is a cutset of $G$; let $C$ be a component of $G - S$. Denote $G_1 = (S \cup C)$, and $G_2 = G - C$. Clearly, $G$ is obtained by subatomic identification from a copy of $G_1$ and a copy of $G_2$. By Theorem 3.7, both $G_i$'s are ptolemaic. Thus, by induction hypothesis, $G_1, G_2 \in \mathcal{C}$. Hence, $G \in \mathcal{C}$. This proves that $\mathcal{P} \subseteq \mathcal{C}$.
Conversely, since complete graphs are ptolemaic, since $\mathcal{P}$ has property (ii) by Theorem 3.7, and since $\mathcal{C}$ is the smallest class of graphs satisfying (i) and (ii), it follows that $\mathcal{C} \subseteq \mathcal{P}$. ■

References