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Induced matchings in intersection graphs

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Abstract

An induced matching in a graph G is a set of edges, no two of which meet a common node or are joined by an edge of G; that is, an induced matching is a matching which forms an induced subgraph. Induced matchings in graph G correspond precisely to independent sets of nodes in the square of the line-graph of G, which we denote by $[L(G)]^2$. Often, if G has a nice representation as an intersection graph, we can obtain a nice representation of $[L(G)]^2$ as an intersection graph. Then, if the independent set problem is polytime-solvable in $[L(G)]^2$, the induced matching problem is polytime-solvable in G. In particular, we show that if G is a polygon-circle graph, then so is $[L(G)]^2$, and the same holds for asteroidal triple-free and interval-filament graphs. It follows that the induced matching problem is polytime-solvable in these classes. Gavril's interval-filament graphs include cocomparability and polygon-circle graphs, and the latter include circle graphs, circular-arc graphs, chordal graphs, and outerplanar graphs.

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A *matching* is a set of edges, no two of which meet a common node. An *induced matching* M in a graph G is a matching such that no two edges of M are joined by an edge of G; that is, an induced matching is a matching which forms an induced subgraph.

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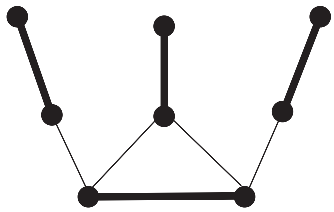

Fig. 1. The bold edges of G form a claw in $[L(G)]^2$.

In this paper, I consider the problem of finding a largest induced matching in a graph G. This problem is NP-hard for bipartite graphs [4,30], and for planar graphs [20].

The *line-graph*, $L(G)$, of graph G has node-set $E(G)$, and an edge joining two nodes exactly when the edges of G they correspond to meet a common node. The *square*, G^2, of graph G has node-set $V(G)$, and two nodes are joined in G^2 exactly when they are joined by an edge or a path of two edges in G. A set of nodes is called *independent* if no two of them are joined by an edge.

Remark 1. For any graph G, every induced matching in G is an independent set of nodes in $[L(G)]^2$, and conversely.

Thus, to find the largest induced matching in a graph G, we can find the largest independent set of nodes in $[L(G)]^2$.

Note that line-graphs are claw-free, and there is a polytime algorithm for finding a largest independent set of nodes in a claw-free graph [24,28]. However, squares of line-graphs need not be claw-free as can be seen from the graph of Fig. 1.

A *strong edge-colouring* of a graph is a partition of its edges into induced matchings, and the *strong chromatic index* of a graph is the minimum size of strong edge-colouring. The *chromatic number* of a graph is the minimum size of a partition of the nodes into independent sets. By Remark 1, it follows that the strong chromatic index of G equals the chromatic number of $[L(G)]^2$.

Given a family $\mathscr{F}$ of non-empty sets, the *intersection graph* $\mathscr{I}(\mathscr{F})$ of $\mathscr{F}$ has node-set $\mathscr{F}$ and an edge between u and v exactly when $u \cap v \neq \emptyset$. There are many well-studied classes of intersection graphs including the following. *Interval graphs* are the intersection graphs of a set of intervals on a line; *chordal graphs* are the intersection graphs of a set of subtrees of a tree; *circular-arc graphs* are the intersection graph of a set of arcs of a circle; *circle graphs* are the intersection graphs of a set of chords of a circle; *polygon-circle graphs* are the intersection graphs of a set of convex polygons inscribed on a circle. Polygon-circle graphs include chordal graphs [18], circular-arc graphs [18], circle graphs, and outerplanar graphs (see [22]).

Cocomparability graphs are the complements of comparability graphs, which are graphs which have a transitive orientation. Cocomparability graphs can be characterized as the intersection graphs of a set of curves between two parallel lines, L_1 and L_2, in the plane, each curve having one endpoint on L_1 and the other on L_2 [17].

Recently, Gavril [11] has introduced a new class of graphs which he calls interval-filament graphs. A graph is an *interval-filament* graph if it is the intersection graph of a set of curves C (called interval-filaments) in the xy-plane with left endpoint, $l(C)$, and right endpoint, $r(C)$, lying on the x-axis such that C lies in the plane above and within the interval $[l(C), r(C)]$. Interval-filament graphs include polygon-circle graphs and cocomparability graphs [11].

The following idea was used in [4] to prove that if G is chordal, then $[L(G)]^2$ is chordal.

Proposition 1. *Let $G = \mathscr{I}(\mathscr{F})$, be the intersection graph of a family $\mathscr{F}$. Then $[L(G)]^2$ is the intersection graph of the family*

$$\mathscr{F}' = \{u \cup v : u, v \in \mathscr{F}, u \neq v, u \cap v \neq \emptyset\},$$

that is, $[L(G)]^2 = \mathscr{I}(\mathscr{F}')$.

Proof. Let $u \cup v$, $w \cup x \in \mathscr{F}'$. Then

$u \cup v$ and $w \cup x$ are joined by an edge in $\mathscr{I}(\mathscr{F}')$
$\Leftrightarrow$ at least one of u, v intersects at least one of w, x
$\Leftrightarrow$ u or $v \in \{w, x\}$, in which case uv and wx meet a common node in G,
or u, v, w, x are all distinct, and say $v \cap w \neq \emptyset$, in which
case uv, vw, wx forms a path of three edges in G
$\Leftrightarrow$ uv and wx are joined by an edge in $[L(G)]^2$. $\square$

Note that if u and v are two intersecting subtrees of a tree T, then $u \cup v$ is also a subtree of T. Similarly, if u and v are two intersecting intervals on a line L, then $u \cup v$ is also an interval on L, and, if u and v are two intersecting arcs of a circle C, then $u \cup v$ is also an arc of C. These observations, together with Proposition 1 give the following corollaries.

Corollary 1 (Cameron [4]). *If G is a chordal graph, then so is* $[L(G)]^2$.

Corollary 2. *If G is an interval graph, then so is* $[L(G)]^2$.

Corollary 3 (Golumbic and Laskar [15]). *If G is a circular-arc graph, then so is* $[L(G)]^2$.

In Fig. 2, I give an example of a circle graph G for which $[L(G)]^2$ is not a circle graph. It is easily checked and also is proved in [2] that the partial wheel G is a circle graph. However, the subgraph of $[L(G)]^2$ induced by the bold edges of G is the 5-wheel, which is not a circle graph [2], so $[L(G)]^2$ is not a circle graph.

I now prove that for the classes of polygon-circle graphs, cocomparability graphs, and interval-filament graphs, if G is in the class, then so is $[L(G)]^2$.

These results are more difficult than for interval, chordal, and circular-arc graphs, because the union of two intersecting polygons inscribed on a circle is not generally another polygon inscribed on a circle, the union of two intersecting curves between

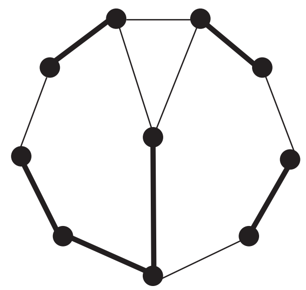

Fig. 2. A circle graph G for which $[L(G)]^2$ is not a circle graph.

two parallel lines is not generally another such curve, and the union of two intersecting interval-filaments is not generally another interval-filament. Thus, we cannot apply Proposition 1 directly. Rather, we show that in each case, if G is the intersection graph of a family $\mathscr{F}$, then $[L(G)]^2$ is the intersection graph of a family $\mathscr{F}'' = \{S(z) : z \in \mathscr{F}'\} = \{S(u \cup v) : u, v \in \mathscr{F},\ u \neq v,\ u \cap v \neq \emptyset\}$, where S has the properties that for $u \cup v,\ w \cup x \in \mathscr{F}'$,

1. $(u \cup v) \cap (w \cup x) \neq \emptyset \Rightarrow S(u \cup v) \cap S(w \cup x) \neq \emptyset$,
2. $S(u \cup v) \cap S(w \cup x) \neq \emptyset \Rightarrow (u \cup v) \cap (w \cup x) \neq \emptyset$.

Then, since $[L(G)]^2$ is the intersection graph of $\mathscr{F}'$, it is also the intersection graph of $\mathscr{F}''$.

Theorem 1. *If G is a polygon-circle graph, then so is $[L(G)]^2$.*

Proof. Let $\mathrm{CH}(z)$ denote the convex hull of z. For $S = \mathrm{CH}$, property (1) above clearly holds since a set is contained in its convex hull. We will prove (2) by showing:

Claim. *If u and v are two intersecting polygons inscribed on a circle C, and if P is another polygon inscribed on C, then $\mathrm{CH}(u \cup v) \cap P \neq \emptyset \Rightarrow [u \cap P \neq \emptyset]$ or $[v \cap P \neq \emptyset]$.*

Then (2) follows by applying the claim twice, once with $P = \mathrm{CH}(w \cup x)$, and then a second time, replacing $\mathrm{CH}(u \cup v)$ with $\mathrm{CH}(w \cup x)$ and replacing P by the one of u and v which intersects $\mathrm{CH}(w \cup x)$. (We know that either u or v does by the first application.)

Proof of the claim. Assume $\mathrm{CH}(u \cup v) \cap P \neq \emptyset$. $\mathrm{CH}(u \cup v)$ and P are convex, so since they intersect, they intersect in a boundary point B of each. Now B is either a vertex of P or lies on an edge E of P. In the first case, B lies on the circle C, and it follows that B must be a vertex of u or of v. In the second case, edge E is a chord of the circle. Since E is a chord of the circle C which intersects $\mathrm{CH}(u \cup v)$ and since u and v are polygons inscribed on C, E must intersect either u or v. □

Note that not all nice subclasses of polygon-circle graphs have the property that for G in the subclass, $[L(G)]^2$ is also in the subclass—as mentioned above, this is not true for circle graphs, and it is not true for the outerplanar graph C_5, the circuit on five nodes, for instance, since $[L(C_5)]^2$ is the complete graph on five nodes, which is not outerplanar.

Theorem 2. *If G is an interval-filament graph, then so is* $[L(G)]^2$.

Proof. Let G be the intersection graph of a set $\mathscr{C}$ of interval-filaments on a horizontal line L. Consider two intersecting members of $\mathscr{C}$, say C_1 and C_2. Consider a plane (multi)graph H, whose nodes are the endpoints and intersection points of C_1 and C_2, and whose edges are the pieces of C_1 and C_2 between these points. All nodes of H except the endpoints of C_1 and C_2 have even degree. Let l^* be the leftmost of the left endpoints of C_1 and C_2, and let l_* be the other left endpoint; and let r^* be the rightmost of the right endpoints of C_1 and C_2, and let r_* be the other right endpoint. Find a path P in H between l_* and r_*. Create a plane graph H' from H by doubling the edges of P in H, (that is, for each edge of P, add a new edge parallel to it), keeping each new edge "close" to the original, so that a copy of a piece of C_i intersects only filaments that C_i does. H' is a plane graph with exactly two nodes of odd degree, l^* and r^*, thus there is a planar Eulerian walk W of H' from l^* to r^*. Let $S(C_1 \cup C_2) = W$. Then it is clear that (1) and (2) hold, and thus $[L(G)]^2$ is also an interval-filament graph. □

This idea can also be used to prove:

Theorem 3 (Golumbic and Lewenstein [16]). *If G is a cocomparability graph, then so is* $[L(G)]^2$.

This result was proved by Golumbic and Lewenstein [16] in a different way. They proved that if G is a k-trapezoid graph, then so is $[L(G)]^2$. Cocomparability graphs are the union over all k, of k-trapezoid graphs.

Proof of Theorem 3. Let cocomparability graph G be the intersection graph of a set $\mathscr{C}$ of curves between two parallel vertical lines, L_1 and L_2, in the plane, each curve having one endpoint on L_1 and the other on L_2. Consider two intersecting members of $\mathscr{C}$, say C_1 and C_2. Consider a plane (multi)graph H, whose nodes are the endpoints and intersection points of C_1 and C_2, and whose edges are the pieces of C_1 and C_2 between these points. All nodes of H except the endpoints of C_1 and C_2 have even degree. Find a path P in H between one of the endpoints on L_1 and one of the endpoints on L_2. Create a plane graph H' from H by doubling the edges of P in H, keeping each new edge "close" to the original, so that a copy of a piece of C_i intersects only curves that C_i does. H' is a plane graph with exactly two nodes of odd degree, and thus there is a planar Eulerian walk W of H', with one endpoint on L_1 and the other on L_2. Let $S(C_1 \cup C_2) = W$. Then it is clear that (1) and (2) hold, and thus $[L(G)]^2$ is also a cocomparability graph. □

Polytime algorithms have been given for finding a largest independent set of nodes in chordal graphs [9], in circular-arc graphs [10], and in interval-filament graphs [11], and one is well known for cocomparability graphs (find a transitive acyclic orientation of the complement [12,13], and then find a largest clique). By Corollaries 1, 3 and Theorems 2, 3, these provide polytime algorithms for finding a largest induced matching in these classes.

Polytime algorithms have also been given for finding a minimum colouring (that is, partition of the nodes into independent sets) of chordal graphs [9], and of cocomparability graphs (see for example, [6] or [5]). Again, by Corollary 1 and Theorem 3, these provide polytime algorithms for finding a minimum strong edge-colouring in these classes. However, this approach does not help in the case of circular-arc graphs (or more generally for polygon-circle or interval-filament graphs), as chromatic number is NP-hard for circular-arc graphs [8]. In [8], a polytime algorithm is given for determining for fixed k, if a circular-arc graph is k-colourable; this then provides a polytime algorithm for determining for fixed k, if a circular-arc graph has a strong edge-colouring with at most k colours.

Gavril's algorithm for finding a largest independent set of nodes in an interval-filament graph requires as input the interval-filament representation. It follows that the approach given by Theorem 2 for finding a largest induced matching in an interval-filament graph also requires such a representation. Given the interval-filament representation of G, a representation of $[L(G)]^2$ can be constructed in polytime as described in the proof of Theorem 2, and then Gavril's algorithm can be applied to find a largest independent set of nodes in $[L(G)]^2$.

So far, the only polytime algorithm known for the independent set problem in polygon-circle graphs is Gavril's algorithm for the more general class of interval-filament graphs. For the other classes of graphs mentioned, namely, chordal graphs, cocomparability graphs, and circular-arc graphs [14], polytime algorithms exist for the independent set problem, and thus for the induced matching problem, which do not require any representation but simply the adjacency information.

An independent set of three nodes is called an *asteroidal triple* (AT) if between each pair in the triple there exists a path that avoids the neighbourhood of the third. A graph is *asteroidal triple-free* (*AT-free*) if it contains no asteroidal triple. AT-free graphs are not yet known to be the intersection graphs of some nice family, however they contain several classes of intersection graphs including interval graphs [23] and cocomparability graphs [7]. Chordal graphs may have asteroidal triples. Note that although powers of AT-free graphs are AT-free [26], line-graphs of AT-free graphs need not be AT-free; Jou-Ming Chang pointed out that K_6, the complete graph on six nodes, is AT-free, but its line-graph is not.

Theorem 4. *If G is AT-free, so is* $[L(G)]^2$.

Theorem 4 was proved independently in [19]. In [3,21], polytime algorithms are given for finding a largest independent set of nodes in AT-free graphs. Thus these algorithms provide polytime algorithms for finding a largest induced matching in AT-free graphs. The complexity of chromatic number in AT-free graphs has not yet been determined.

To prove Theorem 4, I will prove the following.

Proposition 2. *If* $[L(G)]^2$ *has an asteroidal triple, then* G *has an asteroidal triple.*

Proof. Where e is an edge of G, let $v(e)$ denote the corresponding node of $[L(G)]^2$. Suppose $\{v(x), v(y), v(z)\}$ is an AT in $[L(G)]^2$. Let $v(x)=v(e_1), v(e_2), \ldots, v(e_k)=v(y)$ be the nodes of a chordless path P in $[L(G)]^2$, which avoids the neighbourhood of $v(z)$. Since $v(e_i)v(e_{i+1})$ is an edge of $[L(G)]^2$, e_i and e_{i+1} either meet a common node in G or are joined by an edge of G. If three of the e_i's either met a common node in G or formed a path of three edges in G, then the corresponding $v(e_i)$'s would induce a triangle in $[L(G)]^2$, so P would not be a chordless path in $[L(G)]^2$. It follows that the subgraph of G formed by the e_i's consists of paths of one or two edges. Let's call these paths $p_1, p_2, \ldots, p_m$ where p_1 is either e_1 or e_1e_2 and the others follow in order as in P. There must be an edge of G between p_j and p_{j+1}; choose one such edge for each j and call them the *suppressed edges.*

Now, if $p_j = e_ie_{i+1}$ is a path with two edges, and u is its middle node, then there cannot be an edge f of G from u to p_{j+1}, because then where $e_k \subseteq p_{j+1}$ meets f, $v(e_k)$ is joined to both $v(e_i)$ and $v(e_{i+1})$ in $[L(G)]^2$, so P would have a chord. Thus the first node of p_j is joined to p_{j-1} and the last node of p_j is joined to p_{j+1} (unless p_j is the first or last path).

Now, if $p_j = e_i$ is a path with one edge, then e_i is met by two suppressed edges, one joining it to p_{j-1} and one joining it to p_{j+1}. These two suppressed edges may meet either one or both ends of e_i.

Let P' consist of all suppressed edges, the two-edge paths p_j, and the one-edge paths p_h both of whose ends are met by suppressed edges, and x and y (if not already listed). P' is a path in G starting with $x=e_1$ and ending with y. Note that every node of P' is met by some e_i.

Let x' be an end of x, y' an end of y, and z' an end of z. I claim that $\{x', y', z'\}$ is an AT in G. Since $\{v(x), v(y), v(z)\}$ is independent in $[L(G)]^2$, $\{x', y', z'\}$ is independent in G. P' contains an $x'y'$-path P'' in G. P'' does not contain any neighbours of z' in G because if node v of P'' is a neighbour of z' in G, and v meets edge e_i of P', then $v(e_i)$ and $v(z)$ are joined in $[L(G)]^2$, contradicting the choice of P. It follows that $\{x', y', z'\}$ is an AT in G. □

In [4], a set N of edges in a graph G was defined to be *neighbourly* if every pair of edges of N either meet a common node or are joined by an edge of G. For any graph G, neighbourly sets of edges correspond precisely to cliques in $[L(G)]^2$. Polytime algorithms have been given for finding a largest clique in chordal graphs [9], circular-arc graphs [10], interval filament graphs [11], and in cocomparability graphs (for example, [6] or [5]), and thus by Corollaries 1, 3 and Theorems 2, 3, these provide polytime algorithms for finding a largest neighbourly set in these classes. Also, polytime algorithms have been given for finding a minimum partition of the nodes of a graph into cliques (that is, to find a minimum clique cover) in chordal graphs [9] and circular-arc graphs [10], and is well-known for cocomparability graphs (find a transitive acyclic orientation of the complement [12,13], and then find a minimum colouring), so these provide polytime algorithms for finding a minimum partition of the edges of a graph in these classes into neighbourly sets (that is, to find a minimum neighbourly set cover).

An NC-algorithm is one which uses polynomially many parallel processors and whose running time is polynomial in the logarithm of the length of the input. As noted in [4], it is straightforward to design an NC algorithm to find $[L(G)]^2$ for graph G. Thus, as noted in [4], the NC algorithms in [25] provide NC algorithms for finding a maximum induced matching, minimum strong edge-colouring, maximum neighbourly set, and minimum neighbourly set cover in chordal graphs. Similarly, NC algorithms for maximum independent set [1,29], maximum clique [1], and minimum clique cover [29] in circular-arc graphs provide NC algorithms for maximum induced matching, maximum neighbourly set, and minimum neighbourly set cover in this class. Also, the NC algorithm for minimum clique cover in cocomparability graphs [27] provides an NC algorithm for minimum neighbourly set cover in this class.

Please note that when I have mentioned a polytime or NC algorithm for a problem, I have generally only mentioned the first such algorithm found. Often, improved algorithms also exist.

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