A decomposition theorem for chordal graphs and its applications

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Fundamental objects to play with

**Definition**

A graph is chordal iff it has no chordless cycle of length $\geq 4$.

**Maximal Cliques**

under inclusion

**Minimal Separators**

A subset of vertices $S$ is a minimal separator if $G$ if there exist $a, b \in G$ such that $a$ and $b$ are not connected in $G - S$. and $S$ is minimal for inclusion with this property.
An example

3 minimal separators \{b\} for f and a, \{c\} for a and e and \{b, c\} for a and d.
Maximal Clique trees

A maximal clique tree (sometimes clique tree) is a tree $T$ that satisfies the following three conditions:

- Vertices of $T$ are associated with the maximal cliques of $G$.
- Edges of $T$ correspond to minimal separators.
- For any vertex $x \in G$, the cliques containing $x$ yield a subtree of $T$. 

The following statements are equivalent and characterize chordal graphs:

(i) $G$ has a simplicial elimination scheme
(ii) Every minimal separator is a clique
(iii) $G$ admits a maximal clique tree.
(iv) $G$ is the intersection graph of subtrees in a tree.
(v) Any MNS (LexBFS, LexDFS, MCS) provides a simplicial elimination scheme.
An example
**Clique tree**

*Clique tree* of $G = a$ minimum size tree model of $G$

for a clique tree $T$ of $G$:

- vertices of $T$ correspond precisely to the maximal cliques of $G$
- for every maximal cliques $C, C'$, each clique on the path in $T$ from $C$ to $C'$ contains $C \cap C'$
- for each edge $CC'$ of $T$, the set $C \cap C'$ is a *minimal separator* (an inclusion-wise minimal set separating two vertices)

Note: we label each edge $CC'$ of $T$ with the set $C \cap C'$. 

![Diagram of a clique tree](image)
the *clique graph* \( C(G) \) of \( G \) = intersection graph of maximal cliques of \( G \)
the reduced clique graph $C_r(G)$ of $G = \text{graph on maximal cliques of } G$ where $CC'$ is an edge of $C_r(G) \iff C \cap C'$ is a minimal separator.
Combinatorial structure of $C_r(G)$

**Lemma 1: M.H and C. Paul 95**

If $C_1, C_2, C_3$ is a cycle in $C_r(G)$, with $S_{12}, S_{23}$ and $S_{23}$ be the associated minimal separators then two of these three separators are equal and included in the third.

**Lemma 2: M.H. and C. Paul 95**

Let $C_1, C_2, C_3$ be 3 maximal cliques, if $C_1 \cap C_2 = S_{12} \subseteq S_{23} = C_2 \cap C_3$ then it yields a triangle in $C_r(G)$
Lemma 3: Equality case

Let $C_1, C_2, C_3$ be 3 maximal cliques, if $S_{12} = S_{23}$ then:

- either the $C_1 \cap C_3 = S_{13}$ is a minimal separator
- or the edges $C_1 C_2$ and $C_2 C_3$ cannot belong together to a maximal clique tree of $G$. 
First properties of reduced clique graphs
Theorem 1 (Gavril 87, Shibata 1988, Blayr and Payton 93)

The clique trees of \( G \) are precisely the maximum weight spanning trees of \( C(G) \) where the weight of an edge \( CC' \) is defined as \(|C \cap C'|\).

Theorem 2 (Galinier, Habib, Paul 1995)

The clique trees of \( G \) are precisely the maximum weight spanning trees of \( C_r(G) \) where the weight of an edge \( CC' \) is defined as \(|C \cap C'|\).

Moreover, \( C_r(G) \) is the union of all clique trees of \( G \).
Applications

- Maximal Cardinality Search can be seen as a Prim algorithm for computing maximal spanning tree of $CS(G)$.
- LexBFS as an example of Maximal Inclusion Search.
Others simplicial elimination scheme

1. Maximal cardinality search
2. Minimal inclusion search (MIS)
3. Others? How to generate all simplicial elimination schemes?
**Maximal Cardinality Search: ** MCS

**Données:** Un graphe $G = (V, E)$ et un sommet source $s$

**Résultat:** Un ordre total $\sigma$ de $V$

Affecter l’étiquette 0 à chaque sommet

$\text{label}(s) \leftarrow \{n\}$

**pour $i \leftarrow n$ à 1 faire**

Choisir un sommet $v$ d’étiquette maximum

$\sigma(i) \leftarrow v$

**pour chaque sommet non-numéroté $w \in N(v)$ faire**

$\text{label}(w) \leftarrow \text{label}(w) + 1$

fin

fin
**Maximal Inclusion Search: MIS**

**Données:** Un graphe $G = (V, E)$ et un sommet source $s$

**Résultat:** Un ordre total $\sigma$ de $V$

Affecter l’étiquette $\emptyset$ à chaque sommet

$\text{label}(s) \leftarrow \{ n \}$

pour $i \leftarrow n$ à $1$ faire

Choisir un sommet $v$ d’étiquette maximale

$\sigma(i) \leftarrow v$

pour chaque sommet non-numéroté $w \in N(v)$ faire

$\text{label}(w) \leftarrow \text{label}(w) \cup \{ i \}$

fin

fin
Theorem

G is chordal graph iff any MIS produces a simplicial elimination scheme.
How to implement MIS search in linear time?

- 2 known linear implementations: LexBFS, MCS
Simplicial elimination schemes

1. Choose a maximal clique tree $T$
2. While $T$ is not empty do
   Select a vertex $x \in F - S$ in a leaf $F$ of $T$;
   $F \leftarrow F - x$;
   If $F = S$ delete $F$;
Canonical simplicial elimination scheme

1. Choose a maximal clique tree $T$
2. While $T$ is not empty do
   Choose a leaf $F$ of $T$;
   Select successively all vertices in $F - S$
   delete $F$;
Does ther exist other simplicial elimination scheme?
Let $G = (V, E)$ be a chordal graph.

$G$ admits at most $|V|$ maximal cliques and therefore the tree is also bounded by $|V|$ (vertices and edges).

But some vertices can be repeated in the cliques. If we consider a simplicial elimination ordering the size of a given maximal clique is bounded by the neighbourhood of the first vertex of the maximal clique.

Therefore any maximal clique tree is bounded by $|V| + |E|$.
Considering a star on \( n \) vertices, shows \( |\mathcal{CS}(G)| \in O(n^2) \)

Not linear in the size of \( G \)
$CS(G)$ is not chordal!
CS(G) is not chordal!
In fact $CS(G)$ is dually chordal (almost chordal) and $CS(CS(G))$ is chordal.
Characterisation Theorem for interval graphs

(0) \( G = (V, E) \) is interval graph.

(i) \( G \) has ...

(ii) It exists a total ordering \( \tau \) of the vertices of \( V \) s.t. \( \forall x, y, z \in G \) with \( x \leq_\tau y \leq_\tau z \) and \( xz \in E \) then \( xy \in E \).

(iii) \( G \) has a maximal clique path. (A maximal clique path is just a maximal clique tree \( T \), reduced to a path).

(iv) \( G \) is the intersection graph of a family of intervals of the real line.
To recognize an interval graph, we just have to compute a maximal clique tree and check if it is a path?
Introduction

Clique trees and clique graphs

First properties of reduced clique graphs

Interval graphs

Decomposition and split minors

Conclusion
Many linear time algorithms already proposed for interval graph recognition ....
using nice algorithmic tools:
graph searches, modular decomposition, partition refinement, PQ-trees ...
Linear time recognition algorithms for interval graphs

- Booth and Lueker 1976, using PQ-trees.
- Korte and Mohring 1981 using LexBFS and Modified PQ-trees.
- Hsu and Ma 1995, using modular decomposition and a variation on Maximal Cardinality Search.
- Corneil, Olariu and Stewart SODA 1998, using a series of LexBFS.
- M.H, McConnell, Paul and Viennot 2000, using LexBFS and partition refinement on maximal cliques.
Other classes

- Path graphs = Intersection graphs of paths on a tree.
- Directed path graphs = Intersection graphs of directed paths on a rooted tree.
Path graphs are in between interval graphs and chodal graphs.

Find a good linear algorithm to recognize path graph is still a research problem.
For an interval graph, its PQ-tree represents all its possible models and can be taken as a canonical representation of the graph (for example for graph isomorphism).

But even path graphs are isomorphism complete. Therefore a canonical tree representation is not obvious for chordal graphs.

$C_r(G)$ is a Pretty Structure to study chordal graphs.
To prove structural properties of all maximal clique trees of a given chordal graph.
Properties of reduced clique graph
• Any $C_r(G)$ graph can be decomposed using multipartite split operations
• Each clique tree uses exactly $k - 1$ edges of the multipartite split
• A clique tree of $G$ is connected in each component $C_i$
Split minors

An edge $e$ in $C_r(G)$ is permissive, if in all triangles containing $e$ the two other edges have the same label.

3 reduction rules

L1 If $v$ is an isolated vertex, remove $v$
L2 If $e$ is a permissible edge contract $e$
L3 If all the edges of the split $X \cup Y$ have the same label, delete edges between $X$ and $Y$

Definition

$H$ is a split-minor of $C_r(G)$, if $H$ can be obtained from $C_r(G)$ using L1, L2 and L3.
Theorem

Every $C_r(G)$ is totally decomposable with the operations L1, L2 and L3.
Asteroidal number

**Definition**

For a graph $G$, a set $A$ of vertices is **asteroidal**, if for each $v \in A$, $A - v$ belongs to one connected component of $G - N(v)$. The **asteroidal number** $a(G)$ is the size of the maximum asteroidal set in $G$.

Computing $a(G)$ is NP-hard for planar graphs but polynomial for HDD-free graphs Kloks, Krastch, Muller 1997.
Theorem M.H., J. Stacho 2009

For a chordal graph $a(G) < k$ iff no labeled k-star is a split-minor of $C_r(G)$

$G$ is interval iff no labeled claw is a split-minor of $C_r(G)$
leafage $l(G)$ of a chordal graph $G = \text{minimum number of leaves in the host tree of a tree model of } G$ \cite{Lin, McKee, West 1998}

**LEAFAGE**

**Input**: chordal graph $G$ and integer $k$

**Query**: decide whether $l(G) \leq k$ (i.e., there exists a tree model of $G$ whose host tree has at most $k$ leaves)

- $l(G) \leq 2$ if and only if $G$ is interval $\Rightarrow O(n + m)$ algorithm \cite{Booth, Lueker 1975}
- $l(G) \leq 3 \Rightarrow O(n^2 m)$ time algorithm \cite{Prisner 1992}
- $l(G) \leq k$ for fixed $k \Rightarrow O(n^{O(k)})$ algorithm

We show: $O(n^3)$ time algorithm for LEAFAGE (allowing arbitrary $k$)
Applications

If $l(G) = k$, an optimal model provides a good implicit representation. Max clique, coloration, ... in $O(k.n)$. $l(G)$ measures the distance from $G$ to an interval graph.
Theorem M.H., J. Stacho 2009

\( l(G) \) can be polynomially computed in \( O(n^3) \) using \( C_r(G) \).

1. Use tokens in the multipartite splits (corresponding to half edges) and propagate them
2. Construct augmenting paths in an associated directed graph preserving the degrees of the tree.
Conclusion

Results:

- A decomposition of reduced clique graphs
- A characterization for asteroidal number using split minors
- Polynomial time algorithm for computing the leafage of chordal graphs (¼ minimizing the number of leaves in a tree representation)
- Extends interval graph recognition
- Based on augmenting paths (similar to those of the matching theory)
Open problems:

- Improve complexity (Linear time?)
- Min-max characterization (for certification)
- Matroid intersection characterization
- Complexity of “vertex leafage” = minimizing the number of leaves in the subtrees of tree models of $G$ (e.g., max #leaves, total #leaves)
### Results so far on $C_r(G)$ as a labelled graph

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<thead>
<tr>
<th></th>
<th>Maximum w. Hamilton Path</th>
<th>Leafage</th>
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<tbody>
<tr>
<td>$G$ arbitrary</td>
<td>NP-complete</td>
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<tr>
<td>Labelled $C_r(G)$</td>
<td>linear</td>
<td>polynomial</td>
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<td></td>
<td>Interval graph recognition</td>
<td>$O(n^3)$</td>
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Thank you for your attention!