2012 Graph Sandwich Problems and other Related Topics

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Welcome to Jeopardy!
Mathematicians of the 20th Century

This 20th century French mathematician wrote the zeroth book on graph theory in 1926.

Who was André Sainte-Laguë?

"Les réseaux (ou graphes)", Paris (1926)
Graph Sandwich Problems

Guessing and Filling-in Missing Edges

Dealing with Partial Information

missing data and deducing consistency
Take your favorite graph property $\Pi$

The $\Pi$ graph sandwich problem asks:

Given:  
- a vertex set $V$
- a mandatory edge set $E^1$
- a larger edge set $E^2$ $(E^1 \subseteq E^2)$

Is there a (sandwich) graph $G = (V,E)$ with $E^1 \subseteq E \subseteq E^2$ that satisfies $\Pi$?

Optional edges: $E^0 = E^2 - E^1$
Forbidden edges: $E^3 = V \times V - E^2$

Remark.
The Classical Recognition Problem is the case where $E^2 = E^1$
nothing optional!
Example

**A Chordal Graph Sandwich**

chordal graph: every cycle of length $\geq 4$ has a chord

$G^1$

$G$

$G^2$
Early references on Graph Sandwich Problems:

M.C. Golumbic, R. Shamir, J. ACM 1993
   Interval Graphs (NP-complete)

H.L. Bodlaender, M.R. Fellows and T.J. Warnow, ICALP 1992
   Chordal Graphs (NP-complete)

M.C. Golumbic, H. Kaplan, R. Shamir, J. Algorithms 1995
   Permutation, Comparability, Circle, … (NP-complete)
   Split, Threshold, Cographs, … (Polynomial)
Golumbic, Kaplan, Shamir, J. Algorithms 1995
Trapezioiod Graph Sandwich Problem

- Recognition is Polynomial
  
  (Ma and Spinrad [1994], Langley [1995])

- Sandwich is NP-Complete
  
  You won’t find this in the literature

- Reduction uses the Betweenness Problem
  
  Similar to the proof for permutation graphs, cocomparability graphs, and unit interval graphs

  Golumbic, Kaplan, Shamir [1995]
A small hierarchy

cocomparability graphs
function graphs
ribbon graphs

trapezoid graphs

parallelogram graphs

permutation graphs
A graph \( G = (V, E) \) is called a \textit{threshold graph} if there exist positive weights \( a_i \ (i \in V) \) and a threshold \( t > 0 \) such that

\[
S \subseteq V \text{ is a stable set } \iff \sum_{s \in S} a_s \leq t.
\]
Threshold graphs (Chvátal & Hammer 1977)

Theorem 1.17. The following are equivalent.

(i) $G$ is a threshold graph.

(ii) $\overline{G}$ is a threshold graph.

(iii) There exist positive weights $w_i \ (i \in V)$ and a threshold $\theta > 0$ such that $xy \in E \iff w_x + w_y > \theta$.

(iv) Repeatedly removing either a universal or an isolated vertex from $G$ results eventually in the empty set.

(v) $G$ does not contain any of $P_4, C_4$ or $2K_2$ as an induced subgraph.

Universal: adjacent to everyone else
Isolated: adjacent to no one
The Structure of Threshold Graphs

isolated vertices

universal vertex
(after isolates are removed)

Nested neighborhoods

Stable Set

Clique
The Structure of Threshold Graphs

- **Stable Set**
- **Clique**

- **isolated vertices**
- **universal vertex** (after isolates are removed)

**Nested neighborhoods**
Threshold Graph Sandwich Problem

Linear time algorithm the Sandwich Problem
Golumbic, Kaplan, Shamir [1995]

Given a sandwich instance: \( G^1 = (V, E^1) \) and \( G^2 = (V, E^1 \cup E^0) \)

Greedy Algorithm:

Remove isolated vertices from \( G^1 \)
Remove universal vertices from \( G^2 \)
… repeat … until empty
EXERCISES:

• **Trees:**
  
  The Tree Sandwich Problem is Linear

• **Caterpillars:**

  The Caterpillar Sandwich Problem is NP-Complete

• **Planar Graphs:**

  The Planar Graph Sandwich Problem is just Recognition that $G^1$ is Planar
The Chain Graph Sandwich Problem

Joint work with

Simone Dantas, Celina M. H. de Figueiredo, Sulamita Klein and Frederic Maffray

Chain Graphs

$2K_2$ - free bipartite graphs

$2K_2$:

The forbidden subgraph characterizing chain graphs.
Example

Not a chain graph

Here is a $2K_2$

Here is another $2K_2$
The **chain graph sandwich problem** asks:

- a vertex set \( V \)
  - a mandatory edge set \( E^1 \)
  - a larger edge set \( E^2 \)

Is there a graph \( G = (V, E) \) such that \( E^1 \subseteq E \subseteq E^2 \) with \( G \) being a **chain graph**?

(i.e., a \( 2K_2 \) - free bipartite graph)
We must “break” every $2K_2$ and not create any new $2K_2$ by adding “optional” edges.”
New Result:

The chain graph sandwich problem is NP-complete.

This result stands in contrast to

1) the case where \( E^1 \) is a connected graph, (linear-time)
2) the threshold graph sandwich problem, (linear-time)
3) the chain probe graph problem (polynomial-time)
Theorem (Hammer, Peled, Simeone; others)

Let $G = (X,Y,E)$ be a bipartite graph. The following are equivalent:

1) $X$ can be ordered so that neighborhoods are nested.

2) $Y$ can be ordered so that neighborhoods are nested.

3) $G$ is $2K_2$-free.

4) Every induced subgraph of $G$ has at most one non-singleton component, and has a universal $x$ or a universal $y$ vertex.

5) The vertices of $G$ can be assigned weights $w(v)$ and a threshold $t$ can be assigned such that

$$ (x,y) \in E \text{ if and only if } |w(x) - w(y)| > t, \text{ for all } x \text{ and } y. $$

Remark: $X$ is ordered by number of neighbors (vertex degree).
Ordering the **Vertices**

Notice what happens when we order the vertices:

The Neighborhoods are Nested!
Three names for the same class of graphs:

Chain graphs  (from the nested neighborhood property)
Difference graphs  (from the weights and threshold)
2$K_2$-free bipartite graphs  (from the forbidden subgraph)
can be obtained immediately from condition 4:

Every induced subgraph of $G$ has at most one non-singleton component, and has a universal $x$ or a universal $y$ vertex.

Greedy Algorithm:

Remove singletons; Remove universal vertices; repeat…

Does this look familiar?
The Relation to Threshold Graphs

A bipartite graph $G = (X,Y,E)$ is a chain graph if and only if filling one side (say $Y$) into a clique gives a threshold graph.

Similarly,

A split graph $G = (S,C,E)$ is a threshold graph if and only if erasing all edges of the clique $C$ gives a chain graph.
The Chain Graph Sandwich Problem (cgs)

INPUT: A graph $G^1 = (V, E^1)$ and a set $E^0$ of optional edges (which may be added the graph)

OUTPUT: Is there a subset $F \subseteq E^0$ such that the graph $H = (V, E^1 \cup F)$ is a chain graph?

**Theorem 1.** If $G^1$ has at most one non-trivial connected component, then the chain graph sandwich problem can be solved in linear time.

**The Algorithm:**

Verify that $G^1$ is bipartite and create the bipartition $V = X \cup Y$.

Repeatedly, remove either (1) an isolated vertex in $G^1$ or (2) a universal vertex in $G^2 = (X, Y, E^1 \cup E^0)$. 
The assumption of “connectedness” is crucial for the linear-time complexity – since the “bipartition” of $V$ into $X$ and $Y$ is determined.

**Theorem 2.** The chain graph sandwich problem is **NP-complete**.

Moreover, it remains **NP-complete** even in the case when $G^1$ is a matching.

-- Dantas, de Figueiredo, Golumbic, Klein, Maffray (2010)
since we can exhibit a graph that is a possible solution of the problem, and
check in linear time that it is a chain graph.
NP-Completeness Proof

Reduction from the NP-complete problem:

NOT-ALL-EQUAL MONOTONE 3-SATISFIABILITY
(NAE MONO 3-SAT)

**Instance:** A Boolean function $f$ where each clause consists of exactly three unnegated literals.

**Question:** Is there a truth assignment of the variables such that each clause of $f$ has at least one true literal and at least one false literal?

This problem is equivalent to 3-uniform hypergraph bicoloring and is NP-complete, as proved by Kratochvíl and Tuza [2002].
Consider an instance of **NAE MONO 3-SAT**

Let $f$ have variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.

We associate with $f$ the following instance of **CGS**.

For each variable $x_i$:

- 4 vertices $a_i$, $b_i$, $c_i$, $d_i$
- with two mandatory edges (solid)
- and two forbidden edges (dotted)

In addition, the edges $c_i \ a_{i+1}$ and $d_i \ b_{i+1}$ are forbidden

(for $i = 1, \ldots, n-1$)
For each clause $C = x_i x_j x_k$:

- 6 vertices $u_i^C$, $v_i^C$, $u_j^C$, $v_j^C$, $u_k^C$, $v_k^C$

with

- three mandatory edges (solid)
- three forbidden edges (dotted)

In addition, for each variable $x_i$ in clause $C$, the four forbidden edges

$a_i u_i$, $b_i u_i$ and $c_i v_i$, $d_i v_i$
For each clause $C = x_i x_j x_k$:

6 vertices $u_i^C, v_i^C, u_j^C, v_j^C, u_k^C, v_k^C$

with

three mandatory edges (solid)
three forbidden edges (dotted)

In addition, for each variable $x_j$ in clause $C$,
the four forbidden edges
$a_j u_j$, $b_j u_j$ and $c_j v_j$, $d_j v_j$
For each clause $C = x_i x_j x_k$:

6 vertices $u^C_i, v^C_i, u^C_j, v^C_j, u^C_k, v^C_k$

with

three mandatory edges (solid)
three forbidden edges (dotted)

In addition, for each variable $x_k$ in clause $C$, the four forbidden edges $a_k u_k, b_k u_k$ and $c_k v_k, d_k v_k$
Lemma 1. (⇒) If there exists an NAE-satisfying truth assignment for \( f \), then the corresponding instance of CGS \((V, E^1, E^2)\) admits a chain graph \( H = (V,E) \) such that \( E^1 \subseteq E \subseteq E^2 \).

Lemma 2. (⇐) If the instance \((V, E^1, E^2)\) of CGS admits a chain graph sandwich solution, then there exists an NAE-satisfying truth assignment for \( f \).
Lemma 1: NAE $\rightarrow$ CGS

Partition the vertices into Left and Right:
For each variable $x_i$ in clause $C$,

If $x_i = 1$

If $x_i = 0$
Lemma 1: NAE $\rightarrow$ CGS

Partition the vertices into Left and Right:
For each variable $x_i$ in clause $C$,

If $x_i = 1$

If $x_i = 0$
Lemma 1: NAE $\rightarrow$ CGS

For each clause $C = x_i x_j x_k$:

If $x_i x_j x_k = (1,0,0)$

The variable gadget looks like this.
Lemma 1: NAE $\rightarrow$ CGS

For each clause $C = x_i x_j x_k$:

If $x_i x_j x_k = (1,0,0)$

The variable gadget looks like this.
Lemma 1: NAE $\Rightarrow$ CGS

For each clause $C = x_i x_j x_k$:

- If $x_i x_j x_k = (1, 0, 0)$

  The variable gadget looks like this.

  Then, we refine the partition for the clause gadget.

- We do something similar for $x_i x_j x_k = (1, 1, 0)$
Lemma 1: NAE \( \rightarrow \) CGS

For each clause \( C = x_i x_j x_k \):

If \( x_i x_j x_k = (1,0,0) \)

Add Optional edges to the Required edges –

We then prove this is possible, and gives a chain graph.
Lemma 2: NAE $\leftarrow$ CGS

For the opposite direction, given a chain sandwich

$$G = (L,R, E^1 \cup F)$$

we first prove that

- $F$ contains exactly one of the optional (diagonal) edges of each variable gadget, and this defines our truth assignment.

then prove that

- each filled-in clause gadget insures that at least one variable is positive and one negative.

Q.E.D.

For details, see the paper in *Annals of Operations Research*
Discrete Mathematics in the late 20th Century

This once bearded algorithmic graph theorist introduced the class of Tolerance Graphs at the 13th Southeastern Conference in 1982.

Who is Martin Charles Golumbic?

"A generalization of interval graphs with tolerances", (with C.L. Monma)
The Tolerance Graph Sandwich Problem is NP-complete
The Matrix View of Sandwiches

- Adjacency matrix: \{0, 1, *\} entries, where * means optional or don’t know or don’t care.
Matrix Sandwich Problems

- The consecutive ones matrix sandwich problem and the circular ones matrix sandwich problem are NP-complete.
  Golumbic and Wassermann [1996]

- The Ferrers matrix sandwich problem can be solved in $O(mn)$ time.
  Golumbic [1996]
Matrix Sandwich Problems, cont.

**Theorem.** The rectangular block decomposition sandwich problem can be solved in $O(mn)$ time.

**Theorem.** Square block sandwich problem is NP-complete.

Fig. 1. A rectangular block pattern.
The Matrix View of Sandwiches

- Matrix sandwich problem (0/1 completion)
- Hypergraph sandwich problems
- Boolean function completion problems
- Poset sandwich problems
- **Peanut Butter** Sandwich Problems
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<th>Date</th>
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<tr>
<td>Fri, March 4</td>
<td>10AM - 6PM</td>
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<tr>
<td>Sat, March 5</td>
<td>10AM - 6PM</td>
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<td>Sun, March 6</td>
<td>10AM - 4PM</td>
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March is National Peanut Month!
Partitioned Probe Graphs

a special case of the Sandwich Problem

All optional edges are concentrated within one independent set $N$, i.e., $E^0 = N \times N$.

Example: The Probe Game
• Take and interval graph.
• Choose a subset of vertices $N$.
• Erase the edges in $N \times N$.
• Give this Probe Problem to your students to solve.
(Fill in the missing edges.)
Think of $N \times N$ as a hole in the sandwich graph that needs to be completed.
New Result of Bang Le (2010)

The partitioned probe graph problem is polynomial-time solvable for chain graphs.
Many Known Results on Partitioned Probe Problems

- Partitioned Interval Probe is polynomial
  Johnson and Spinrad [2001], McConnell and Spinrad [2001]
- Partitioned Chordal Probe is polynomial
  Berry, Golumbic, Lipshteyn [2006]
- Partitioned THIS and THAT Probe is polynomial
  Whom and Ever [2008-11]
All optional edges are concentrated within a set $N$ but $G^1$ is not totally zero on $N$.

This was actually the problem that we first solved for Chordal Graph Completion and then for Chordal Probe.
DEFINITION: A **chain probe graph** is a bipartite graph $G = (V,E)$ for which their exists an independent set $S$ of vertices and a set $F$ of pairs of vertices in $S \times S$ such that the graph $H = (V, E \cup F)$ is a chain graph.

Not a chain graph

A chain graph
Theorem (Golumbic, Maffray, Morel, 2008)

A bipartite graph is a chain probe graph if and only if it is \{C_6, P_7, 3K_2, H_1, H_2, H_3\}-free.

Let’s see why
No good, still has a $2K_2$

Adding an edge spoils bipartite
No good, still has a $2K_2$

Similarly for the other forbidden subgraphs.

Proof of sufficiency see the paper.
Details of the Converse

please see our 2008 paper in the

*Annals of Operations Research*
Theorem (Golumbic, Maffray, Morel, 2008)

Recognizing (non-partitioned) chain probe graphs can be done in $O(n^2)$ time.
<table>
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<tr>
<th>Chain Graph Recognition</th>
<th>Chain Graph Partitioned Probe</th>
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<td>Linear</td>
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<td>Chain Graph Non-Partitioned Probe</td>
<td>Chain Graph Sandwich</td>
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<tr>
<td>$O(n^2)$</td>
<td>NP-Complete</td>
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</table>
Two groups of high school students, the Girls of Evelina and the Boys of Hartman, visited the Hecht museum on the same day.

1. The students came on their own, arriving and leaving at different times, but

2. Each school gathered their students together to tour as a group.

3. The Hecht museum is rather small, so if a girl and a boy were in the museum at the same time, they certainly met each other.

   Adina's boyfriend is Eitan and her brother is Doron.

   Yael's boyfriend is Doron and her brother is Eitan.

Is it possible that both girls met their boyfriends, but not their brothers?
A Solution

The answer to this questions is, \"No\", which is rather easy to see.

If Adina did not meet Doron, it follows that their time periods in the museum did not overlap. So let's suppose Adina came and left and afterwards Doron came and left (the reverse case is similar.)
Since Yael and Adina toured together with the Evelina girls, their time periods overlapped. Therefore, if Yael met Doron, her time interval in the museum must have totally spanned the gap between the time Adina left and the time Doron arrived.

Now, what about Eitan?
On the one hand, Eitan toured together with Doron, so the boys' intervals overlapped.

On the other hand, if Eitan met Adina, then his interval would also have to totally span the gap between Adina's leaving time and Doron's arriving time.

But then Eitan would have been in the museum with his sister Yael and seen her.

This would be a contradiction.

So it is impossible that both girls met their boyfriends and not their brothers.
This story is one example of “reasoning about intervals”, and it can be modeled by a graph $G$.  

- one vertex for each student  
- connect two vertices by an edge if the associated two students “did not meet” in the museum, that is, their time intervals were **disjoint**.
In our story,

- the **girls** form a set $X$ with no edges between them, and
- the **boys** form a set $Y$ with no edges between them.
The Common X and Y Intervals

Intersection of all girls

Intersection of all boys

Extreme cases:
The Common X and Y Intervals

Intersection of all girls
Intersection of all boys

Extreme cases:

If these two BIG intervals intersect, then $G$ has no edges.
The Common X and Y Intervals

Intersection of all girls

Intersection of all boys

Extreme cases:

If no blue intersects a red (e.g., morning/afternoon), then $G$ has all possible edges (i.e., complete).
Forbidden $2K_2$

The Adina-Doron-Yael-Eitan “proof” shows that the graph $G$ cannot contain the configuration $2K_2$

$2K_2$: The forbidden subgraph characterizing chain graphs.

So these are the chain graphs, and we have returned home again.
THANK YOU