Edge Intersection Graphs of Single Bend Paths on a Grid

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Joint work with: Marina Lipshteyn and Michal Stern (2007)
Additional results: Andrei Asinowski, Bernard Ries and Andrew Suk (2008)
and New results by: Therese Biedl and M. Stern (2009)
Daniel Heldt, Kolja Knauer and Torsten Ueckerdt (2010)
Motivation

VLSI layout problem has two phases:

1. **Placement**: a circuit has a set of nets, each net represented by terminals on the planar layout (grid) connected by a wire (path). The wires are edge-disjoint but not vertex-disjoint.

2. **Wiring**: convert two-dimensional edge-disjoint layout into three-dimensional vertex-disjoint layout. A vertical connection between levels is called a via and is used at each level change.
Edge Intersection Graphs of Paths on a Grid (EPG)

Graph $G = (V, E)$

Paths on a grid $\Gamma$

Each vertex $v$ in $V(G)$ corresponds to a path $P_v$ in $\Gamma$ such that $(x, y) \in E(G) \iff$ paths $P_x$ and $P_y$ share at least one edge in $\Gamma$
First Results

Theorem. Every graph has an EPG representation on an $n \times 2n$ grid.

Figure 3: The graph $G$ and an EPG representation of $G$. 
First Results, cont.

Theorem. Every graph has a monotone EPG representation on an $n \times (n+e)$ grid.
So we will look at a specific, interesting special subclass of EPG

Restrict the number of bends allowed in a path.
Bends of Wires on a Layout

- In chip manufacturing, each bend requires a transition hole.
- High number of such holes may increase the layout area and the cost of the chip.
- A lot of research is done to minimize the number of bends in a layout.
Single Bend: $B_1$-EPG

Definition: Only one bend (per path)

Theorem: Every tree is $B_1$-EPG.
Single Bend: \( B_1 \)-EPG

Definition: Only one bend (per path)

Theorem: Every tree is \( B_1 \)-EPG

Figure 5: A \( B_1 \)-EPG representation of a tree.
Some Examples of the Chordless Cycle $C_4$

True Pie

False Pie

$C_4$

Various types of Frames
Some Examples of the Chordless Cycle $C_4$

True Pie

False Pie

$C_4$

Various types of Frames
The Structure of Representations of Cycles

The case of $C_4$:

**Theorem:** Let $P$ be a $B_1$-EPG representation of a chordless 4-cycle $C_4$.

Then $P$ is either a true pie or a false pie or a frame.
Representations of Longer Cycles

The case of $C_n \ (n > 4)$:

**Theorem:** Chordless cycles of all sizes have $B_1$-EPG representations; they all use an even number of bends.
Anti-holes

• The graphs $\overline{C}_5$ and $\overline{C}_6$ are $B_1$-EPG

Bernard Ries (2009): The remaining anti-holes $\overline{C}_n$ ($n \geq 7$) are not $B_1$-EPG

Proof. Three cases: true pie, false pie and frames on $\{1, n-1, 2, n-2\}$. 
The Complete Bipartite Graphs

- The graph $K_{3,3}$ has no $B_1$-EPG representation
- For any $m$, $K_{m,\infty}$ is $B_{2m-2}$

Asinowski and Suk (2008):

- $K_{2,n}$ is $B_1$ if and only if $n \leq 4$
  (hence $K_{2,5}$ is not $B_1$)
- $K_{m,\infty}$ is not $B_{2m-3}$
- $K_{m,n}$ is $B_{\max\{\lceil m/2 \rceil, \lceil n/2 \rceil\}}$ (hence $K_{m,m}$ is $B_{m/2}$)
- more to say about $K_{m,n}$ later ...
Theorem. Cliques have exactly two possible $B_1$-EPG representations:

- **“edge-clique”**: all paths share a common grid edge
- **“claw clique”**: each path contains two of the three grid edges of a claw

Example: Representations of $K_6$
**Theorem:**

Let $\mathbf{P}$ be a $\mathbf{B}_1$-EPG representation of $G$. Every clique in $G$ corresponds to either an edge-clique or a claw-clique.

**Proof.** If no path bends, we have an interval graph.

If some $\mathbf{P}_u$ bends, using legs $e_1$ and $e_2$, then all paths that share only $e_1$ or $e_2$ must share some other edge – namely a common leg $e_3$. 
The Helly Property for $B_1$-EPG Graphs

Single bend paths on a grid do **NOT** satisfy the Helly property.

Example: \[ P_1 \cap P_2, P_1 \cap P_3, P_2 \cap P_3 \neq \emptyset \]

But \[ P_1 \cap P_2 \cap P_3 = \emptyset \]
But $B_1$-EPG do have the following property:

**Theorem:** Let $\mathcal{P}$ be a $B_1$ representation with $P_1, \ldots, P_m \in \mathcal{P}$.

(1) If $P_i$ and $P_j$ share an edge $\forall i, j$, then $P_1 \cap \ldots \cap P_m \neq \emptyset$.

AND Strong Helly Number 4 of $B_1$ representations:

(2) There exist paths $P_i, P_j, P_k, P_l$ such that

$$P_i \cap P_j \cap P_k \cap P_l = P_1 \cap \ldots \cap P_m$$

This is best possible:

(any three intersect in a point, but the intersection of all four is empty.)
The Hermann Grid
**Branch Graphs**

**Definition:** Let $C$ be a subset of vertices of $G$. The branch graph $B(G/C)$:

- **vertices:** $V - C$
- **edges:** $(x,y)$ is an edge in $B(G/C)$ iff
  1. $(x,y) \notin E(G)$ and
  2. $x,y$ have a common neighbor in $C$, and both have private neighbors in $C$. 

![Diagram of a branch graph](image)
Example. The 4-sun $S_4$:

Choosing the clique $K_4$
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Choosing the clique $K_4$, the branch graph $B(S_4/K_4)$ is the chordless cycle $C_4$. 

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Example. The graph $A_4$ (below):
Choosing the clique $K_6$, the branch graph $B(A_4/K_6)$ is the clique $K_4$. 

Branch Graphs
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Branch Graphs
Theorem: Let $K$ be a maximal clique of a $B_1$-EPG graph $G$.

1. If $K$ is an edge-clique, then the branch graph $B(G/K)$ can be 2-colored.
2. If $K$ is a claw-clique, then $B(G/K)$ can be 3-colored.

Corollary. The graph $A_4$ is not $B_1$-EPG.

Proof: $B(A_4/K_6)$ requires 4 colors.
More Forbidden Subgraphs

These graphs are not $B_1$-EPG, but they are $B_2$-EPG.

- $K_{3,3}$ (Golumbic, Lipshteyn, Stern)
- $K_{2,5}$ (Asinowski)
- $W_k$ ($k \geq 5$) (Ries)
**$B_k$-EPG Graphs Hierarchy**

- $B_0$-EPG graphs $\equiv$ interval graphs
  
  $C_4$ is not interval, but is $B_1$-EPG,

  so $B_0$-EPG $\subset B_1$-EPG

  $K_{3,3}$ is not $B_1$-EPG, but is $B_2$-EPG,

  so $B_1$-EPG $\subset B_2$-EPG

**QUESTION:**

Is $B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_k \subset B_{k+1} \cdots$?

Andrei Asinowski proved this for odd $k$.

Heldt, Knauer and Ueckerdt have proven this for all $k$. 
More Motivation
Open Questions

**Characterization:**

- What are the $B_1$-EPG graphs?
- What are the $B_k$-EPG graphs? ($k = 2, 3, \ldots$)

**Monotonic Representations:**

The 4-wheel $W_4$ is not “monotonic-$B_1$”

but it is $B_1$-EPG  (Bernard Ries)

So $\text{monotonic-}B_1 \subset B_1$-EPG

- Is $\text{monotonic-}B_k \subset B_k$-EPG? ($k = 2, 3, \ldots$)
Computational Complexity

Theorem (Heldt, Knauer and Ueckerdt (2010))

Recognizing $B_1$-EPG graphs is NP-complete.

Open Complexity Questions

- Coloring of $B_1$-EPG graphs
- Finding maximum independent sets
The bend number of a graph

• What is the minimum number of bends to represent a given graph $G$? That is,

Define the **bend number**

$$b(G) \equiv \text{the smallest } k, \text{ such that } G \text{ has a } B_k\text{-EPG representation}$$
The bend number of a graph

<table>
<thead>
<tr>
<th>b(G)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>trees</td>
<td>1</td>
</tr>
<tr>
<td>planar</td>
<td>( \leq 5 )</td>
</tr>
<tr>
<td>outerplanar</td>
<td>2</td>
</tr>
<tr>
<td>bipartite planar</td>
<td>2</td>
</tr>
<tr>
<td>line graphs</td>
<td>2</td>
</tr>
<tr>
<td>( K_{m,n} ) (( m \leq n ))</td>
<td>( \leq \left\lceil \frac{n}{2} \right\rceil ) *</td>
</tr>
<tr>
<td></td>
<td>( \geq 2m - 2 ) **</td>
</tr>
</tbody>
</table>

* bound is tight for \( m=n \)  
  Heldt, K. Knauer and Ueckerdt

** bound is tight for \( n \geq m^4 - 2m^3 + 5m^2 - 4m \)  
  Heldt, K. Knauer and Ueckerdt
A Different Famous Grid

$S_{\delta^0 k_u}$
Yet More Open Questions

Several minimization problems:

- What is the minimal size of a grid for specific $B_k$-EPG graphs?

Theorem: The minimum size of a universal grid – one that can represent every graph on $n$ vertices – is $\Theta(n^2)$.

Proof: (upperbound): Our original $n \times 2n$ grid construction
(lowerbound): Pach&Suk observation: $K_{n/2,n/2}$ requires a grid with $n^2/4$ edges.

- Does Monotonic-$B_1$-EPG require larger grid size than an unconstrained $B_1$-EPG representation?
- By how much?
The Story Began on Trees
Bell Labs in New Jersey (Spring 1981)

John Klincewicz: Suppose you are routing phone calls in a tree network. Two calls interfere if they share an edge of the tree. How can you optimally schedule the calls?
The Story Begins

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An Olive Tree Network
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An Olive Tree Network

• A call is a path between a pair of nodes.
• A typical example of a type of intersection graph.
• Intersection here means “share an edge”.
• Coloring this intersection graph is scheduling the calls.
(EPT-Graphs) Edge Intersection Graphs of **Paths** in a Tree

Golumbic and Jamison [1985]

Each vertex $v$ in $V(G_{EPT})$ corresponds to a path $P_v$ in $T$.

$$(x,y) \in E_{EPT} \iff \text{paths } P_x \text{ and } P_y \text{ intersect on at least one edge in } T.$$
Known Results on EPT Graphs

Theorem (Golumbic & Jamison, 1985):

\[ \text{chordal } \cap \text{ EPT } \equiv \text{ deg3Tree EPT} \]

where \textit{chordal} means no induced \( C_m \) for \( m \geq 4 \).

Theorem (Golumbic, Lipshteyn & Stern, 2005):

\[ \text{weakly chordal } \cap \text{ EPT } \equiv \text{ deg4Tree EPT} \]

where \textit{weakly chordal} means no induced \( C_m \) and no induced \( \overline{C}_m \) for \( m \geq 5 \).
What about chordal $B_1$-EPG?

Results by Andrei Asinowski and Bernard Ries (2009)

**Theorem 1.**
Every claw-free, chordal graph is $B_1$-EPG.

**Theorem 2.**
Every diamond-free, chordal graph is $B_1$-EPG.

**Theorem 3.**
Every split graph with maxclique at most 3 is $B_1$-EPG.
What about chordal $B_1$-EPG?

Results by Andrei Asinowski and Bernard Ries (2009)

**Theorem 4.**

A bull-free, chordal graph is $B_1$-EPG if and only if every vertex neighborhood $N(v)$ is $T_2$-free.
Conclusion

We have presented the families of $B_k$-EPG graphs and posed many Open Questions – especially for the class $B_1$-EPG.

We hope it will be a source of much interesting work in the coming years.
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More on Algorithmic Graph Theory
Thank You!