Image Processing - Lesson 6

## Fourier Transform - Part II

- Discrete Fourier Transform - 1D
- Discrete Fourier Transform - 2D
- Fourier Properties
- Convolution Theorem
- FFT
- Examples

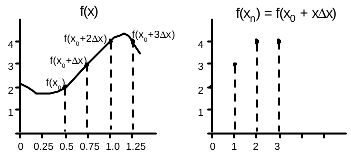
## Discrete Fourier Transform

Move from f(x) ($x \in R$) to f(x) ($x \in Z$) by sampling at equal intervals.

$f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[n-1]\Delta x),$

Given N samples at equal intervals, we redefine f as:

$f(x) = f(x_0+x\Delta x) \quad x = 0, 1, 2, \ldots, N-1$




## Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is defined as:

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{\frac{-2piux}{N}} \qquad u = 0, 1, 2, ..., N-1$$

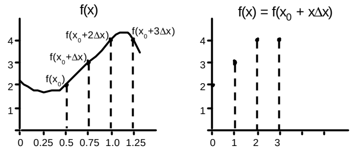
Matlab: F=fft(f);

The Inverse Discrete Fourier Transform (IDFT) is defined as:

$$f(x)=\sum_{u=0}^{N-1} F(u)e^{\frac{2piux}{N}} \qquad x = 0, 1, 2, ..., N-1$$

Matlab: F=ifft(f);

## Discrete Fourier Transform - Example




$F(0) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi i 0x}{4}} = 1/4 \sum_{x=0}^{3} f(x)\, 1$

$= 1/4(f(0) + f(1) + f(2) + f(3)) = 1/4(2+3+4+4) = 3.25$

$F(1) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i\pi/2}+4e^{-\pi i}\, 4e^{-i3\pi/2}] = \frac{1}{4}[-2+i]$

$F(2) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-4\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i\pi}+4e^{-2\pi i}\, 4e^{-3\pi i}] = \frac{-1}{4}[-1-0i] = \frac{-1}{4}$

$F(3) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-6\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i3\pi/2}+4e^{-3\pi i}\, 4e^{-i9\pi/2}] = \frac{1}{4}[-2-i]$

Fourier Spectrum:

$|F(0)| = 3.25$

$|F(1)| = [(-1/2)^2 + (1/4)^2]^{0.5}$

$|F(2)| = [(-1/4)^2 + (0)^2]^{0.5}$

$|F(3)| = [(-1/2)^2 + (-1/4)^2]^{0.5}$

## Discrete Fourier Transform - 2D

Image f(x,y) x = 0,1,...,N-1 y= 0,1,...,M-1

The Discrete Fourier Transform (DFT) is defined as:

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y)\, e^{-2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

u = 0, 1, 2, ..., N-1
v = 0, 1, 2, ..., M-1

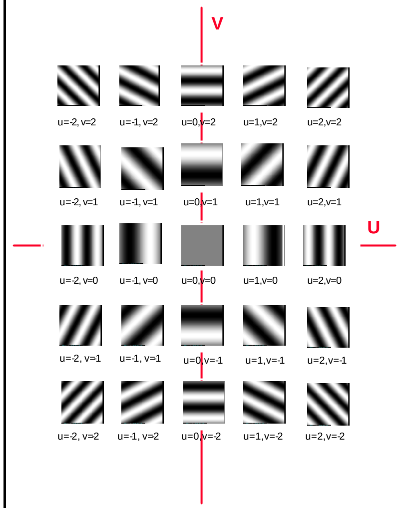
Matlab: F=fft2(f);

The Inverse Discrete Fourier Transform (IDFT) is defined as:

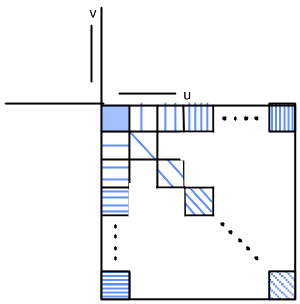
$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u,v)\, e^{2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

x = 0, 1, 2, ..., N-1
y = 0, 1, 2, ..., M-1

Matlab: F=ifft2(f);



## Fourier Transform - Image




## Visualizing the Fourier Transform Image using Matlab Routines

- F(u,v) is a Fourier transform of f(x,y) and it has complex entries.

  F = fft2(f);

- In order to display the Fourier Spectrum |F(u,v)|
  - Cyclically rotate the image so that F(0,0) is in the center:

    F = fftshift(F);

  - Reduce dynamic range of |F(u,v)| by displaying the log:

    D = log(1+abs(F));

**Example**:

|F(u)| = 100 4 2 1 0 0 1 2 4

Cyclic |F(u)| = 0 1 2 4 100 4 2 1 0

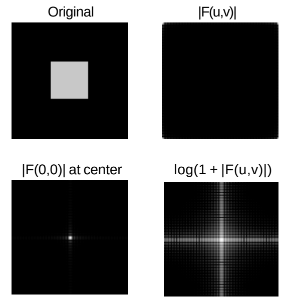
Display in Range([0..10]):

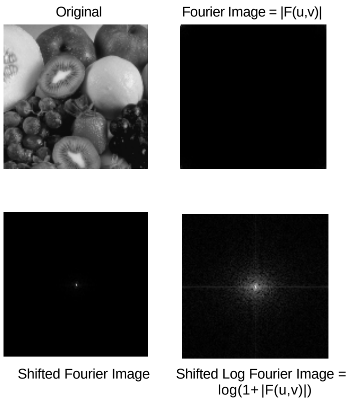
|F(u)|/10 = 0 0 0 0 10 0 0 0 0

log(1+|F(u)|) = 0 0.69 1.01 1.61 4.62 1.61 1.01 0.69 0

log(1+|F(u)|)/0.462 = 0 1 2 4 10 4 2 1 0

## Visualizing the Fourier Image - Example






## Properties of The Fourier Transform

**Distributive (addition)**

$$\tilde{F}[f_1(x,y) + f_2(x,y)] = \tilde{F}[f_1(x,y)] + \tilde{F}[f_2(x,y)]$$

**Linearity**

$$\tilde{F}[a\, f(x,y)] = a\,\tilde{F}[f(x,y)]$$

$$a\, f(x,y) \;\text{———}\; a\, F(u,v)$$

**Cyclic**

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$

$$F(x,y) = F(x+N,y+N)$$

**Symmetric** if f(x) is real:

$$F(u,v) = F^*(-u,-v)$$

thus:

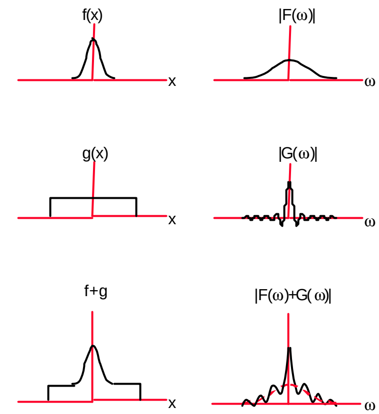
$$|F(u,v)| = |F(-u,-v)|$$

Fourier Spectrum is symmetric

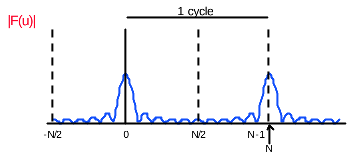
**DC (Average)**

$$F(0,0) = \frac{1}{N}\frac{1}{M}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x)\, e^0$$

## Distributive: $\tilde{F}\{f+g\} = \tilde{F}\{f\} + \tilde{F}\{g\}$



## Cyclic and Symmetry of the Fourier Transform - 1D Example




## Image Transformations

**Translation**

$$f(x-x_0, y-y_0) \longleftrightarrow F(u,v)\, e^{\frac{-2\pi i(ux_0+vy_0)}{N}}$$

$$f(x,y)\, e^{\frac{2\pi i(u_0x+v_0y)}{N}} \longleftrightarrow F(u-u_0, v-v_0)$$

The Fourier Spectrum remains unchanged under translation:

$$|F(u,v)| = \left|F(u,v)\, e^{\frac{-2\pi i(ux_0+vy_0)}{N}}\right|$$

**Rotation**

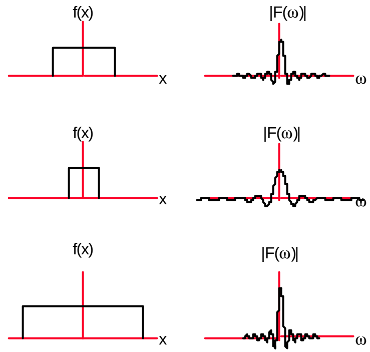
Rotation of f(x,y) by θ ⟷ Rotation of F(u,v) by θ

**Scale**

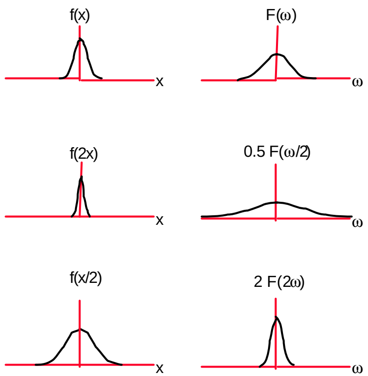
$$f(ax,by) \longleftrightarrow \frac{1}{|ab|} F(u/a, v/b)$$

## Change of Scale- 1D:

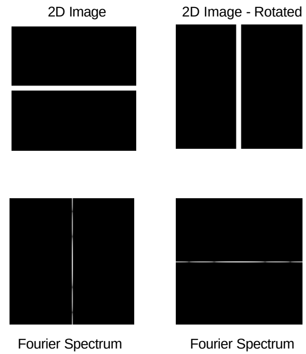
if $\tilde{F}\{f(x)\} = F(w)$ then $\tilde{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{w}{a}\right)$



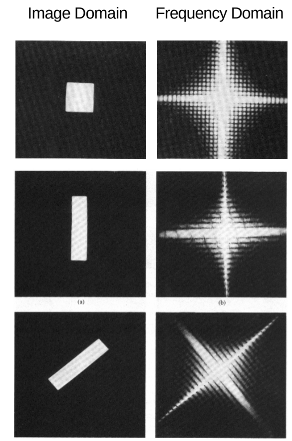
## Change of Scale




## Example - Rotation




## Fourier Transform Examples



## Separabitity

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi i(ux+vy)/n}$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} \left( \sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi ivy/n} \right) e^{-2\pi iux/n}$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} F(x,v)\, e^{-2\pi iux/n}$$

Thus, to perform a 2D Fourier Transform is equivalent to performing 2 1D transforms:

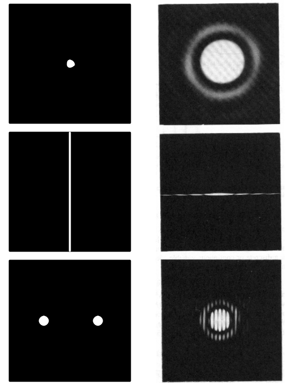
1) Perform 1D transform on EACH column of image f(x,y). Obtain F(x,v).

2) Perform 1D transform on EACH row of F(x,v).

Higher Dimensions:
Fourier in any dimension can be performed by applying 1D transform on each dimension.

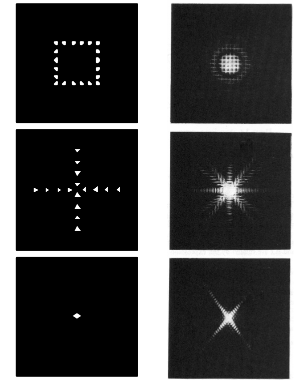
## Fourier Transform Examples

Image Domain | Frequency Domain



## Fourier Transform Examples

Image Domain | Frequency Domain



## Linear Systems and Responses

| | Spatial Domain | Frequency Domain |
|---|---|---|
| Input | f | F |
| Output | g | G |
| Impulse Response | h | |
| Freq. Response | | H |
| Relationship | g=f*h | G=FH |

## The Convolution Theorem

| | |
|---|---|
| g = f * h | g = f h |
| implies | implies |
| G = F H | G = F * H |

Convolution in one domain is multiplication in the other and vice versa

## The Convolution Theorem

$$\tilde{F}\{f(x) * g(x)\} = \tilde{F}\{f(x)\}\tilde{F}\{g(x)\}$$

and likewise

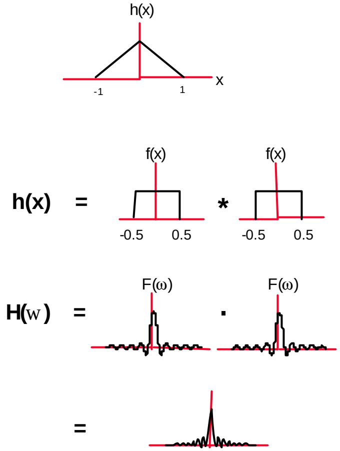
$$\tilde{F}\{f(x)g(x)\} = \tilde{F}\{f(x)\} * \tilde{F}\{g(x)\}$$

f(x,y) * g(x,y) —— F(u,v) G(u,v)

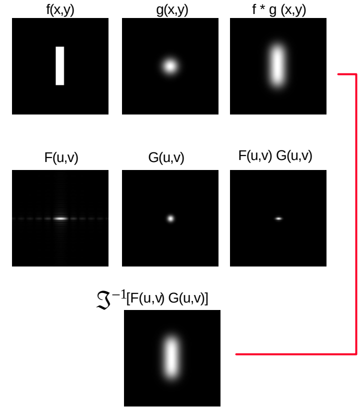
f(x,y) g(x,y) —— F(u,v) * G(u,v)

Convolution in one domain is multiplication in the other and vice versa

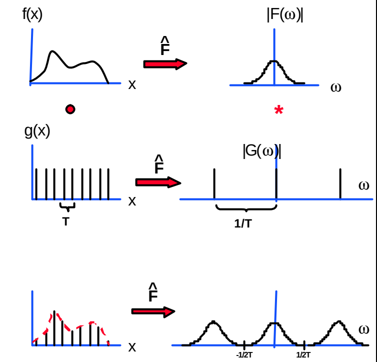
Example:



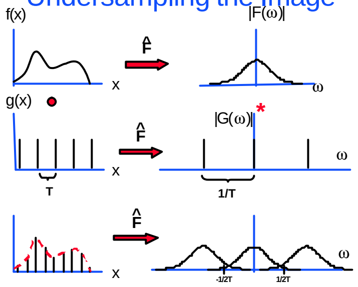

## Convolution Theorem - 2D Example




## Sampling the Image



# Undersampling the Image

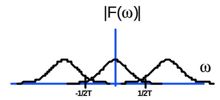



# Critical Sampling

- If the maximal frequency of f(x) is $\omega_{max}$ , it is clear from the above replicas that $\omega_{max}$ should be smaller that 1/2T.
- Alternatively:

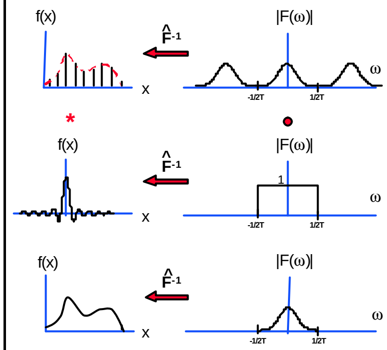
$$\frac{1}{T} > 2w_{max}$$

- **Nyquist Theorem**: If the maximal frequency of f(x) is $\omega_{max}$ the sampling rate should be larger than 2 $\omega_{max}$ in order to fully reconstruct f(x) from its samples.
- If the sampling rate is smaller than 2 $\omega_{max}$ overlapping replicas produce **aliasing** .




# Optimal Interpolation

- It is possible to fully reconstruct f(x) from its samples:



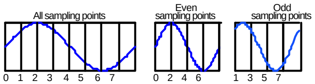
## Fast Fourier Transform - FFT

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-2\pi i u x}{N}} \qquad u = 0, 1, 2, ..., N-1$$

$O(n^2)$ operations

$$F(u)=\underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x)\, e^{\frac{-2\pi i u 2x}{N}}}_{\text{even x}} + \underbrace{\frac{1}{N}\sum_{x=0}^{N/2-1} f(2x+1)\, e^{\frac{-2\pi i u (2x+1)}{N}}}_{\text{odd x}}$$

$$=\frac{1}{2}\left[\underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x)\, e^{\frac{-2\pi i u x}{N/2}}}_{\text{Fourier Transform of of N/2 even points}} + e^{\frac{-2\pi i u}{N}} \underbrace{\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x+1)\, e^{\frac{-2\pi i u x}{N/2}}}_{\text{Fourier Transform of of N/2 odd points}}\right]$$




The Fourier transform of N inputs, can be performed as 2 Fourier Transforms of N/2 inputs each + one complex multiplication and addition for each value i.e. O(N).

Note, that only N/2 different transform values are obtained for the N/2 point transforms.

$$F_N(u)=\frac{1}{2}\left[\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x)\, e^{\frac{-2\pi i u x}{N/2}} + e^{\frac{-2\pi i u}{N}} \frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x+1)\, e^{\frac{-2\pi i u x}{N/2}}\right]$$

$$F_N(u)=\frac{1}{2}\left[F^e_{N/2}(u)+e^{\frac{-2\pi i u}{N}} F^o_{N/2}(u)\right]$$

For u' = u+N/2 : $e^{\frac{-2\pi i u'}{N}} = e^{\frac{-2\pi i (u+N/2)}{N}} = e^{\frac{-2\pi i u}{N}} e^{-\pi i} = -e^{\frac{-2\pi i u}{N}}$

obtain :

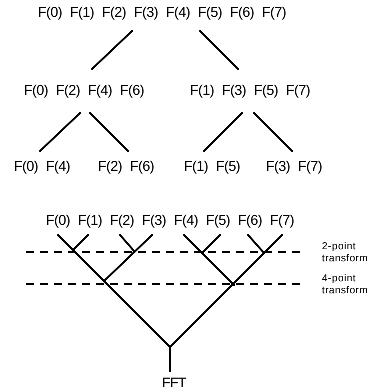
$$F_N(u)=\frac{1}{2}\left[F^e_{N/2}(u)+e^{\frac{-2\pi i u}{N}} F^o_{N/2}(u)\right]$$

For u = 0, 1, 2, ..., N/2-1

$$F_N\left(u+\frac{N}{2}\right)=\frac{1}{2}\left[F^e_{N/2}(u)-e^{\frac{-2\pi i u}{N}} F^o_{N/2}(u)\right]$$

Thus: only one complex multiplication is needed for two terms.

Calculating $F^e_{N/2}(u)$ and $F^o_{N/2}(u)$ is done recursively by calculating $F^e_{N/4}(u)$ and $F^o_{N/4}(u)$.




FFT : $O(n\log(n))$ operations

FFT of NxN Image: $O(n^2\log(n))$ operations

## Frequency Enhancement