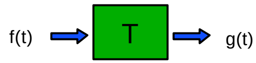
Image Processing - Lesson 4

Introduction to Fourier Transform - Linear Systems

- Linear Systems
  - Definitions & Properties
  - Shift Invariant Linear Systems
  - Linear Systems and Convolutions
  - Linear Systems and sinusoids
  - Complex Numbers and Complex Exponentials
  - Linear Systems - Frequency Response

Linear Systems

- A linear system T gets an input f(t) and produces an output g(t):



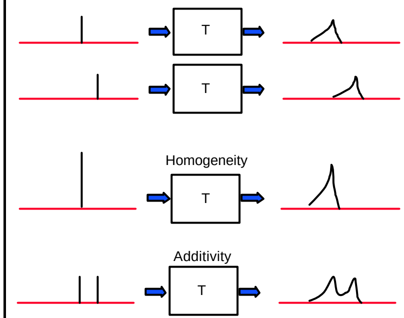
$$g(t)=T\{f(t)\}$$

- In the discrete caes:
  - input : f[n] ,   n = 0,1,2,…
  - output: g[n] ,   n = 0,1,2,…

$$g[n]=T[f(n)]$$

Linear System Properties

- A linear system must satisfy two conditions:
  - Homogeneity: $T\{a f[n]\}=aT\{f[n]\}$
  - Additivity: $T\{f_1[n]+f_2[n]\}=T\{f_1[n]\}+T\{f_2[n]\}$



Linear System - Example

- **Contrast change** by grayscale stretching around 0:

$$T\{f(x)\} = af(x)$$

  - Homogeneity:

$$T\{bf(x)\} = abf(x) = baf(x) = bT\{f(x)\}$$

  - Additivity:

$$\begin{aligned} T\{f_1(x)+f_2(x)\} &= a(f_1(x)+f_2(x)) \\ &= af_1(x)+af_2(x) \\ &= T\{f_1(x)\}+ T\{f_2(x)\} \end{aligned}$$

Linear System - Example

- **Convolution**

$$T\{f(x)\} = f*a$$

  - Homogeneity:

$$T\{bf(x)\} = (bf)*a = b(f*a) = bT\{f(x)\}$$

  - Additivity:

$$\begin{aligned} T\{f_1(x)+f_2(x)\} &= (f_1+f_2)*a \\ &= f_1*a+f_2*a \\ &= T\{f_1(x)\}+ T\{f_2(x)\} \end{aligned}$$

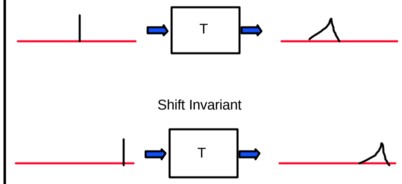
Shift-Invariant Linear System

- Assume T is a linear system satisfying

$$g(t)=T\{f(t)\}$$

- T is a shift-invariant linear system iff:

$$g(t-t_0)=T\{f(t-t_0)\}$$



Shift-Invariant Linear System - Example

- **Contrast change** by grayscale stretching around 0:

$$T\{f(x)\} = af(x) = g(x)$$

  – Shift Invariant :

$$T\{f(x-x_0)\} = af(x-x_0) = g(x-x_0)$$

- **Convolution**:

$$T\{f(x)\} = f(x)*a = g(x)$$

  – Shift Invariant :

$$T\{f(x-x_0)\} = f(x-x_0)*a$$

$$= \sum_i f(i-x_0)a(x-i) = \sum_j f(j)a(x-j-x_0)$$

$$= g(x-x_0)$$

Matrix Multiplication as a Linear System

- Assume f is an input vector and T is a matrix multiplying f:

$$g = Tf$$

- g is an output vector.
- Claim: A matrix multiplication is a linear system:
  – Homogeneity $T(af) = aTf$
  – Additivity $T(f_1+f_2) = Tf_1 + Tf_2$
- Note that a matrix multiplication is not necessarily shift-invariant.

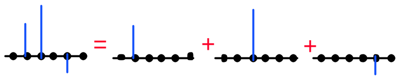
Impulse Sequence

- An impulse signal is defined as follows:

$$d[n-k] = \begin{cases} 0 & \text{where} \quad n \neq k \\ 1 & \text{where} \quad n = k \end{cases}$$

- Any signal can be represented as a linear sum of scales and shifted impulses:

$$f[n] = \sum_{j=-\infty}^{\infty} f[j]d[n-j]$$



Shift-Invariant Linear System is a Convolution

Proof:

- f[n] input sequence
- g[n] output sequence
- h[n] the system impulse response:

$$h[n]=T\{\delta[n]\}$$

$$g[n]=T\{f[n]\}=T\left\{\sum_{j=-\infty}^{\infty} f[j]d[n-j]\right\}$$

$$=\sum_{j=-\infty}^{\infty} f[j]T\{d[n-j]\} \quad (\text{from linearity})$$

$$=\sum_{j=-\infty}^{\infty} f[j]h[n-j] \quad (\text{from shift-inariancce})$$

$$= f*h$$

The output is a sum of scaled and shifted copies of impulse responses.

Convolution as a Matrix Multiplication

- The convolution (wrap around):

$$[1\ 2\ 0\ 0\ -1\ -2]*[3\ 2\ 1]=[6\ 5\ 2\ -3\ -8\ -2]$$

can be represented as a matrix multiplication:

Circulant Matrix

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ -3 \\ -8 \\ -2 \end{bmatrix}$$

- The matrix rows are flipped and shifted copies of the impulse response.
- The matrix columns are shifted copies of the impulse response.

Convolution Properties

- Commutative:

$$T_1 * T_2 * f = T_2 * T_1 * f$$

  - Only shift-invariant systems are commutative.
  - Only circulant matrices are commutative.

- Associative:

$$(T_1 * T_2) * f = T_1 * (T_2 * f)$$

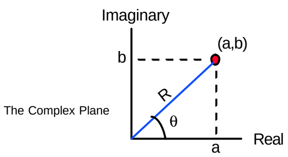
  - Any linear system is associative.

- Distributive:

$$(T_1 + T_2) * f = T_1 * f + T_2 * f$$
$$\text{and } T * (f_1 + f_2) = T * f_1 + T * f_2$$

  - Any linear system is distributive.

Complex Numbers



- Two kind of representations for a point (a,b) in the complex plane
  - The Cartesian representation:

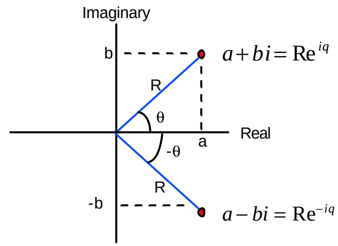
$$Z = a + bi \qquad where \quad i^2 = -1$$

  - The Polar representation:

$$Z = Re^{i\theta}$$ (Complex exponential)

- Conversions:
  - Polar to Cartesian: $Re^{i\theta} = R\cos(\theta) + iR\sin(\theta)$
  - Cartesian to Polar $a + bi = \sqrt{a^2 + b^2}\; e^{i\tan^{-1}(b/a)}$

- Conjugate of Z is Z^*:
  - Cartesian rep. $(a + ib)^* = a - ib$
  - Polar rep. $\left(Re^{i\theta}\right)^* = Re^{-i\theta}$



Algebraic operations:

- addition/subtraction:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

- multiplication:

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

$$Ae^{i\alpha}\; Be^{i\beta} = ABe^{i(\alpha+\beta)}$$

- Norm:

$$|a + ib|^2 = (a + ib)^*(a + ib) = a^2 + b^2$$

$$\left|Re^{i\theta}\right|^2 = \left(Re^{i\theta}\right)^* Re^{i\theta} = Re^{-i\theta}\; Re^{i\theta} = R^2$$

The (Co-) Sinusoid

- The (Co-)Sinusoid as complex exponential:

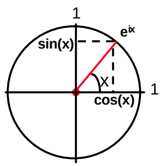
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

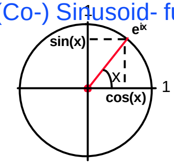
Or

$$\cos(x) = \text{Real}(e^{ix})$$

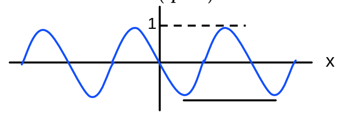
$$\sin(x) = \text{Imag}(e^{ix})$$



The (Co-) Sinusoid- function



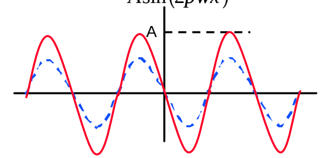
$$\sin(2pwx)$$



- The wavelength of $\sin(2pwx)$ is $\frac{1}{w}$.
- The frequency is w .

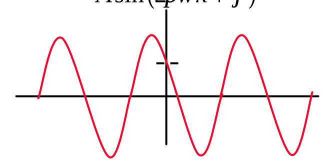
- Changing Amplitude:

$$A\sin(2pwx)$$



- Changing Phase:

$$A\sin(2pwx + j)$$



Scaling and shifting can be represented as a multiplication with Ae^{ij}

$$A\sin(2pwx + j) = \text{Imag}(Ae^{ij}\, e^{i2pwx})$$

Frequency Analysis

- If a function f(x) can be expressed as a linear sum of scaled and shifted sinusoids:

$$f(x) = \sum_{w} F(w) e^{i2pwx}$$

it is possible to predict the system response to f(x):

$$g(x) = T\{f(x)\} = \sum_{w} H(w) F(w) e^{i2pwx}$$

- The Fourier Transform:

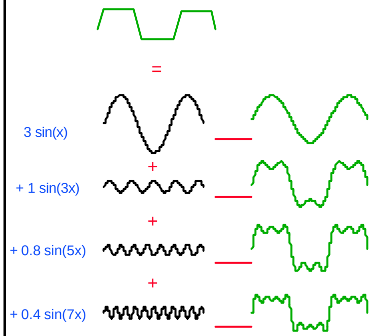
It is possible to express any signal as a sum of shifted and scaled sinusoids at different frequencies.

$$f(x) = \sum F(w) e^{i2pwx}$$

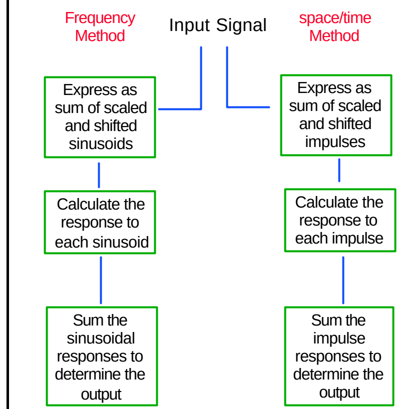
Or

$$f(x) = \int_{w} F(w) e^{i2pwx}\, dw$$

Every function equals a sum of scaled and shifted Sines



Linear System Logic



$$G(w) = F(w)H(w) \qquad g(x) = f(x) * h(x)$$