Linear Systems

- A linear system \( T \) gets an input \( f(t) \) and produces an output \( g(t) \):

\[
f(t) \xrightarrow{T} g(t) = T\{f(t)\}
\]

- In the discrete case:
  - input: \( f[n], \ n = 0, 1, 2, \ldots \)
  - output: \( g[n], \ n = 0, 1, 2, \ldots \)

\[
g[n] = T\{f[n]\}
\]

Linear System Properties

- A linear system must satisfy two conditions:
  - Homogeneity: \( T[af[n]] = aT[f[n]] \)
  - Additivity: \( T[f[n]+g[n]] = T[f[n]] + T[g[n]] \)
Linear System - Example

• Contrast change by grayscale stretching around 0:
  \[ T(f(x)) = af(x) \]
  - Homogeneity:
    \[ T(bf(x)) = abf(x) = baf(x) = bT(f(x)) \]
  - Additivity:
    \[ T(f_1(x) + f_2(x)) = a(f_1(x) + f_2(x)) = af_1(x) + af_2(x) = T(f_1(x)) + T(f_2(x)) \]

Linear System - Example

• Convolution
  \[ T(f(x)) = f*a \]
  - Homogeneity:
    \[ T(bf(x)) = (bf)*a = b(f*a) = bT(f(x)) \]
  - Additivity:
    \[ T(f_1(x) + f_2(x)) = (f_1 + f_2)*a \]
    \[ = f_1*a + f_2*a \]
    \[ = T(f_1(x)) + T(f_2(x)) \]

Shift-Invariant Linear System

• Assume T is a linear system satisfying
  \[ g(t) = T\{f(t)\} \]
  • T is a shift-invariant linear system iff:
    \[ g(t-t_0) = T\{f(t-t_0)\} \]
Shift-Invariant Linear System - Example

• Contrast change by grayscale stretching around 0:
  \[ T(f(x)) = af(x) = g(x) \]
  – Shift Invariant:
  \[ T(f(x-x_0)) = af(x-x_0) = g(x-x_0) \]

• Convolution:
  \[ T(f(x)) = f(x)*a = g(x) \]
  – Shift Invariant:
  \[ T(f(x-x_0)) = f(x-x_0)*a = g(x-x_0) \]

\[
\sum_{j=-\infty}^{\infty} f(x-x_0) \delta(x-j) = g(x-x_0)
\]

Matrix Multiplication as a Linear System

• Assume \( f \) is an input vector and \( T \) is a matrix multiplying \( f \):
  \[ g = Tf \]
  – Homogeneity \( T(af) = aTf \)
  – Additivity \( T(f_1 + f_2) = Tf_1 + Tf_2 \)

• Note that a matrix multiplication is not necessarily shift-invariant.

Impulse Sequence

• An impulse signal is defined as follows:
  \[ \delta[n-k] = \begin{cases} 
  0 & \text{where } n \neq k \\
  1 & \text{where } n = k 
\end{cases} \]

• Any signal can be represented as a linear sum of scales and shifted impulses:
  \[ f[n] = \sum_{j=-\infty}^{\infty} f[j] \delta[n-j] \]
Shift-Invariant Linear System is a Convolution

Proof:
- \( f[n] \) input sequence
- \( g[n] \) output sequence
- \( h[n] \) the system impulse response:
  \[ h[n] = T\{\delta[n]\} \]

\[ x[n] * f[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \] (from linearity)
\[ h[n] = \sum_{k=-\infty}^{\infty} \delta[n-k] \] (from shift-invariance)
\[ h[n] = \sum_{k=-\infty}^{\infty} \delta[n-k] \]

The output is a sum of scaled and shifted copies of impulse responses.

Convolution as a Matrix Multiplication

- The convolution (wrap around):
  \[ f[n] * g[n] = \sum_{k=-\infty}^{\infty} f[k] g[n-k] \]

The circulant matrix can be represented as a matrix multiplication:

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & -1 & -2 & -3 & -8 & -2
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 & 0 & 0 & 0
1 & 0 & 1 & 2 & 3 & 0 & 0 & 0
0 & 0 & 1 & 2 & 3 & 0 & 0 & 0
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0
\end{bmatrix}
\]

- The matrix rows are flipped and shifted copies of the impulse response.
- The matrix columns are shifted copies of the impulse response.

Convolution Properties

- Commutative:
  \[ T_1 * T_2 * f = T_2 * T_1 * f \]
  - Only shift-invariant systems are commutative.
  - Only circulant matrices are commutative.

- Associative:
  \[ (T_1 * T_2) * f = T_1 * (T_2 * f) \]
  - Any linear system is associative.

- Distributive:
  \[ T_1 * f_1 + T_2 * f_2 = T_1 * f + T_2 * f \]
  - Any linear system is distributive.
Complex Numbers

- Complex Plane: 
  - Real
  - Imaginary
  - Two kind of representations for a point (a,b) in the complex plane:
    - The Cartesian representation:
      \[ Z = a + bi \text{ where } i^2 = -1 \]
    - The Polar representation:
      \[ Z = Re^{i\theta} \text{ (Complex exponential)} \]
- Conversions:
  - Polar to Cartesian: \[ R = \sqrt{a^2 + b^2} \]
  - Cartesian to Polar: \[ a + bi = \sqrt{a^2 + b^2} e^{i \cos^{-1}(\frac{a}{\sqrt{a^2 + b^2}})} \]

Algebraic operations:
- Addition/subtraction:
  \[ (a+ib) + (c+id) = (a+c) + (b+d)i \]
- Multiplication:
  \[ (a + ib)(c + id) = (ac - bd) + (bc + ad)i \]
  \[ A e^{i\theta} B e^{i\phi} = AB e^{i(\theta + \phi)} \]
- Norm:
  \[ |a + bi|^2 = a^2 + b^2 \]
  \[ |R e^{i\theta}|^2 = Re^0 \quad Re = Re^0 Re^0 = R^2 \]
The (Co-) Sinusoid

- The (Co-)Sinusoid as complex exponential:
  \[ \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \]
  \[ \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \]

Or

\[ \cos(x) = \text{Real}(e^{ix}) \]
\[ \sin(x) = \text{Imag}(e^{ix}) \]

The (Co-) Sinusoid- function

- The wavelength of \( \sin(2\pi x) \) is \( \frac{1}{\omega} \).
- The frequency is \( \omega \).

- Changing Amplitude:
  \[ A \sin(2\pi \omega x) \]

- Changing Phase:
  \[ A \sin(2\pi \omega x + \phi) \]

Scaling and shifting can be represented as a multiplication with \( Ae^{ix} \):

\[ A \sin(2\pi \omega x + \phi) = \text{Imag}(Ae^{ix} e^{2\pi \omega x}) \]
Frequency Analysis

- If a function $f(x)$ can be expressed as a linear sum of scaled and shifted sinusoids:
  
  $$f(x) = \sum F(\omega) e^{i2\pi \omega x}$$

  it is possible to predict the system response to $f(x)$:

  $$g(x) = T[f(x)] = \sum H(\omega) F(\omega) e^{i2\pi \omega x}$$

- The Fourier Transform:
  It is possible to express any signal as a sum of shifted and scaled sinusoids at different frequencies.

  $$f(x) = \sum F(\omega) e^{i2\pi \omega x}$$

  Or

  $$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{i2\pi \omega x} d\omega$$

---

Linear System Logic

**Frequency Method**

- Input Signal

- Express as sum of scaled and shifted sinusoids

  - Calculate the response to each sinusoid

  - Sum the sinusoidal responses to determine the output

  \[ G(\omega) = F(\omega) H(\omega) \]

  \[ g(x) = f(x) * h(x) \]

**Space/Time Method**

- Input Signal

- Express as sum of scaled and shifted impulses

  - Calculate the response to each impulse

  - Sum the impulse responses to determine the output

---