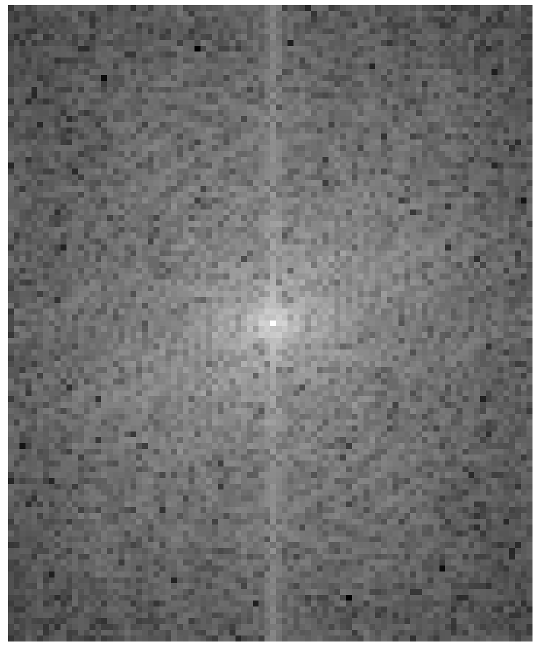
Fourier Transform - Part I

- Introduction to Fourier Transform
  - Image Transforms
  - Basis to Basis
  - Fourier Basis Functions
  - Fourier Coefficients
- Fourier Transform - 1D
- Fourier Transform - 2D

The Fourier Transform



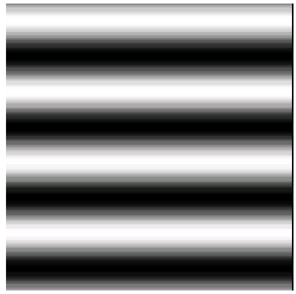
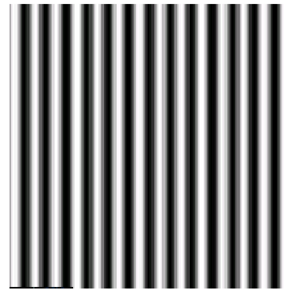
Jean Baptiste Joseph Fourier

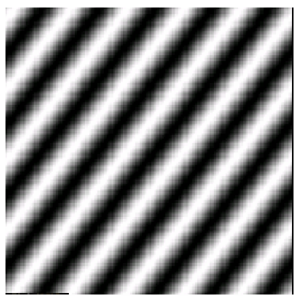
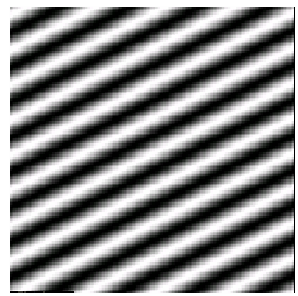
Efficient Data Representation

- Data can be represented in many ways.
- There is a great advantage using an appropriate representation.
- It is often appropriate to view images as combinations of waves.

How can we enhance such an image?

Solution: Image Representation

 = 3  + 5  +

+ 10  + 23  + ...

$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & 8 & 7 \\ 0 & 3 & 5 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ 3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$

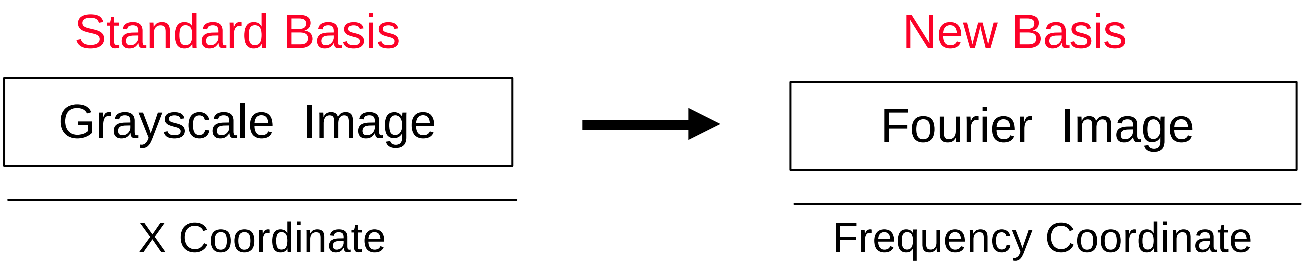
The inverse Fourier Transform

- For linear-systems we saw that it is convenient to represent a signal f(x) as a sum of scaled and shifted sinusoids.

$$f(x) = \int_w F(w) e^{i2pwx}\, dw$$

How is this done?

Transforms: Change of Basis



Standard Basis:

$$[\ a_1\ \ a_2\ \ a_3\ \ a_4\ ] =$$

$$a_1[\ 1\ 0\ 0\ 0\ ] + a_2[\ 0\ 1\ 0\ 0\ ] + a_3[\ 0\ 0\ 1\ 0\ ] + a_4[\ 0\ 0\ 0\ 1\ ]$$

Hadamard Transform:

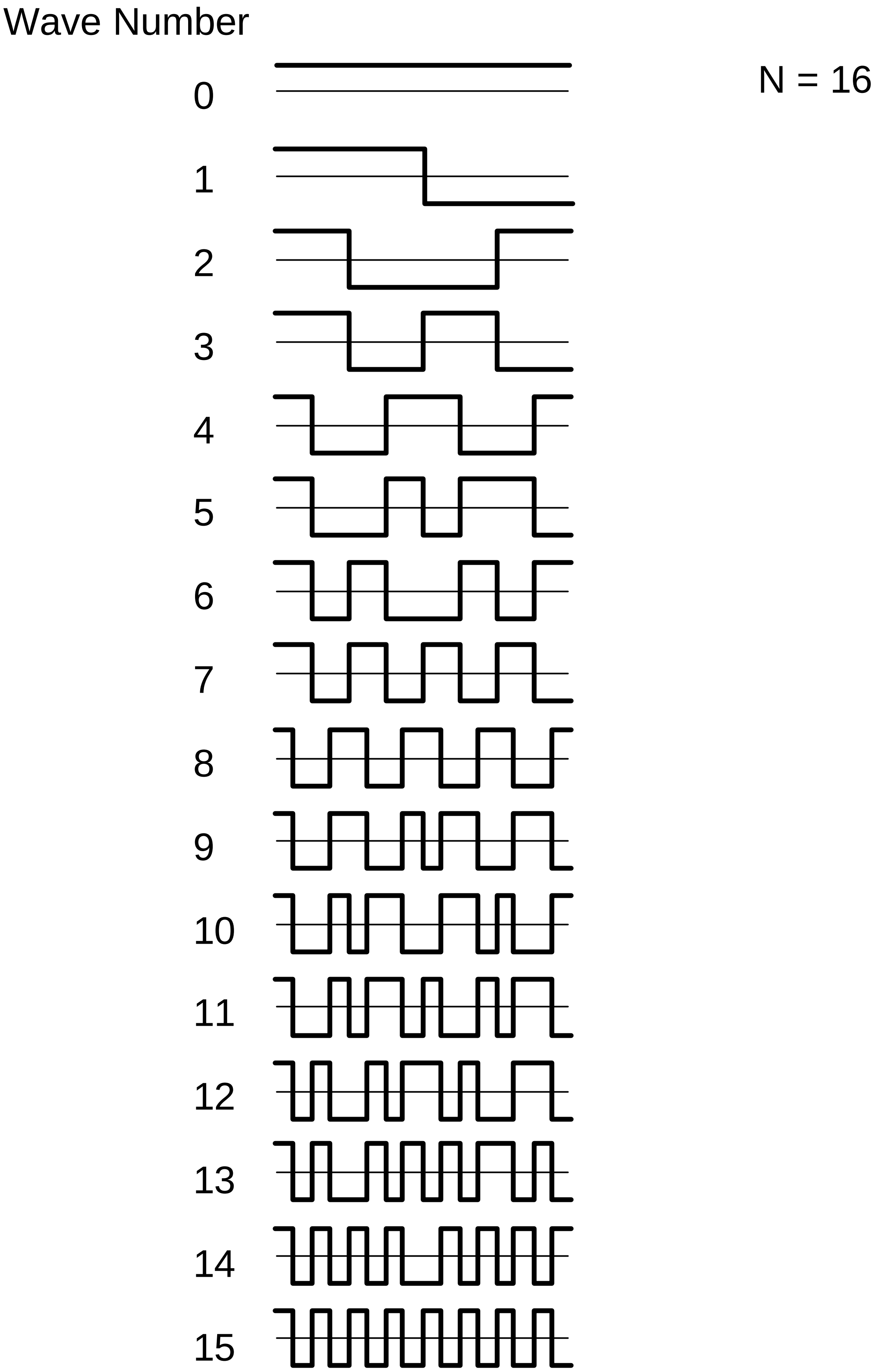
$$[\ 2\ \ 1\ \ 0\ \ 1\ ] =$$

$$= 1[\ 1\ 1\ 1\ 1\ ] + 1/2[\ 1\ 1\ -1\ -1\ ] - 1/2[\ -1\ 1\ 1\ -1\ ] + 0[\ -1\ 1\ -1\ 1\ ]$$

$$= \ [\ 1\ 1/2\ -1/2\ 0\ ]_{\text{Hadamard}}$$

1. Basis Functions.
2. Method for finding the image given the transform coefficients.
3. Method for finding the transform coefficients given the image.

Hadamard Basis Functions - 1D



Finding the transform coefficients

Signal: $\mathbf{X} = [\,2\ 1\ 0\ 1\,]_{standard}$

New Basis:

$\mathbf{T_0} = [\,1\ \ 1\ \ 1\ \ 1\,]$

$\mathbf{T_1} = [\,1\ \ 1\ -1\ -1\,]$

$\mathbf{T_2} = [\,-1\ 1\ \ 1\ -1\,]$

$\mathbf{T_3} = [\,-1\ 1\ -1\ \ 1\,]$

New Coefficients:

$a_0 = \mathbf{\langle X,T_0 \rangle} = \langle [\,2\ 1\ 0\ 1\,], [\,1\ \ 1\ \ 1\ \ 1\,] \rangle /4 = 1$

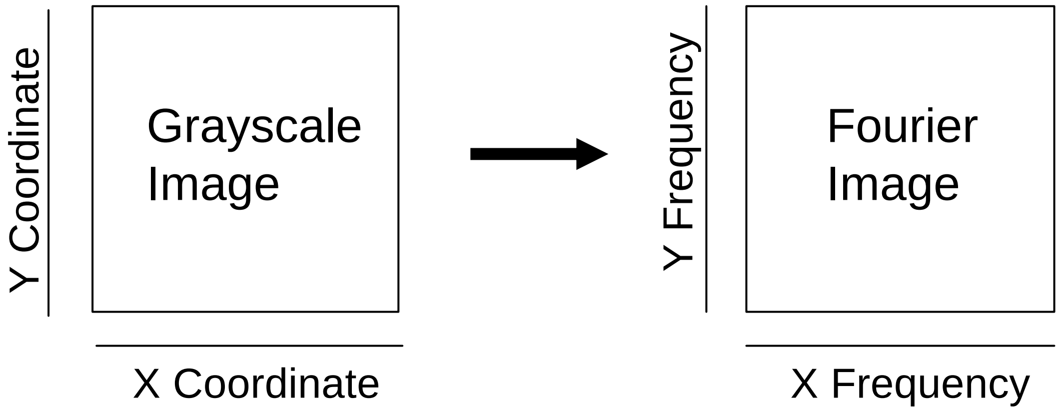
$a_1 = \mathbf{\langle X,T_1 \rangle} = \langle [\,2\ 1\ 0\ 1\,], [\,1\ \ 1\ -1\ -1\,] \rangle /4 = 1/2$

$a_2 = \mathbf{\langle X,T_2 \rangle} = \langle [\,2\ 1\ 0\ 1\,], [\,-1\ \ 1\ \ 1\ -1\,] \rangle /4 = -1/2$

$a_3 = \mathbf{\langle X,T_3 \rangle} = \langle [\,2\ 1\ 0\ 1\,], [\,-1\ \ 1\ -1\ \ 1\,] \rangle /4 = 0$

Signal: $\mathbf{X} = [\,1\ 1/2\ -1/2\ 0\,]_{new}$

Transforms: Change of Basis - 2D



Standard Basis:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hadamard Transform:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 1/2\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - 1/2\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + 0\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 \\ -1/2 & 0 \end{bmatrix}_{\text{Hadamard}}$$

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1. Basis Functions.
2. Method for finding the image given the transform coefficients.
3. Method for finding the transform coefficients given the image.

Standard Basis:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}_{\text{Standard}}$$

coefficients

Basis Elements

Hadamard Transform:

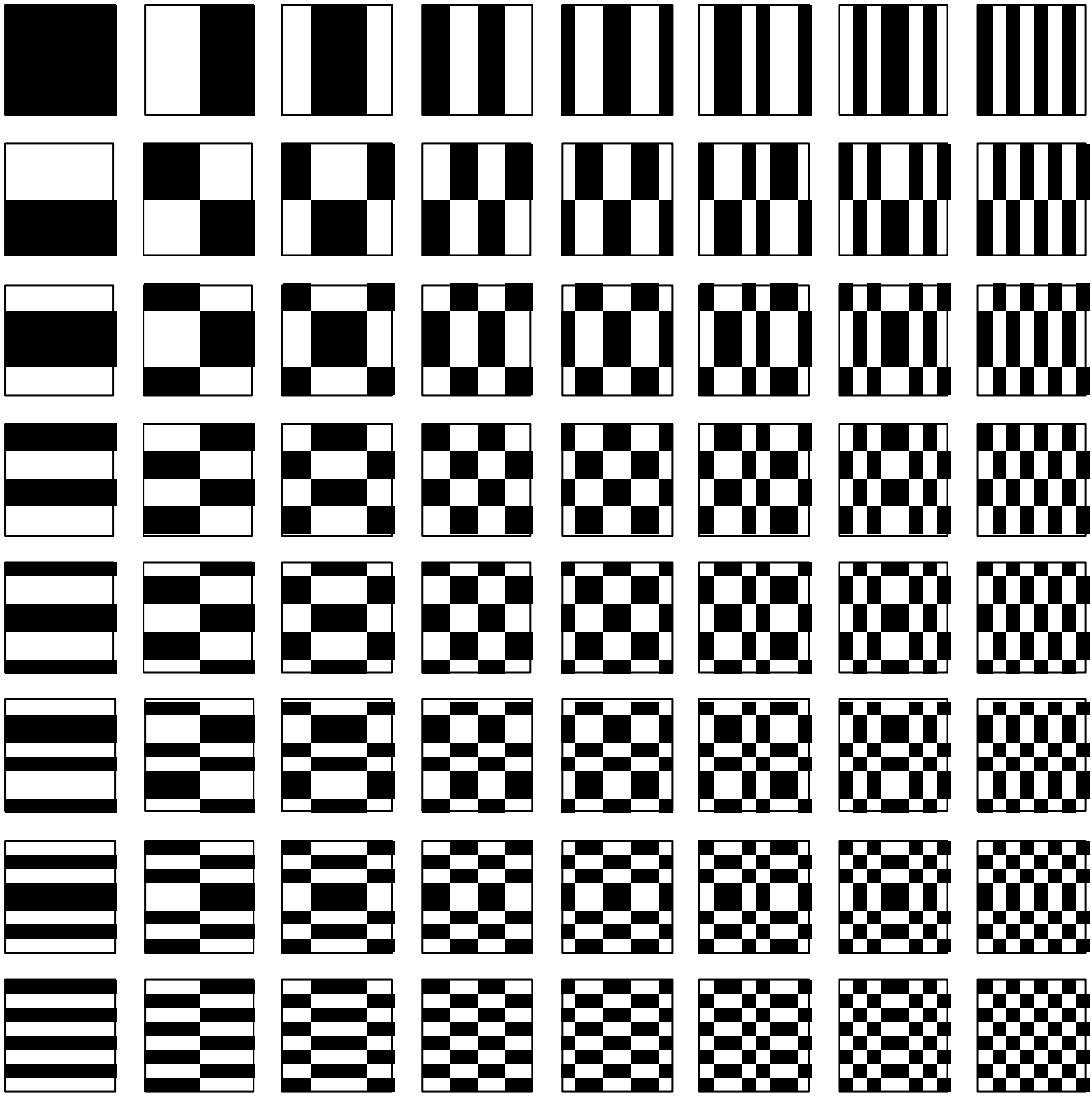
$$\begin{bmatrix} 1 & 1/2 \\ -1/2 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 1/2\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - 1/2\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + 0\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ -1/2 & 0 \end{bmatrix}_{\text{Hadamard}} \longleftrightarrow \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}_{\text{Hadamard}}$$

coefficients

Basis Elements

Hadamard Basis Functions



size = 8x8

Black = +1   White = -1

For continuous images/signals f(x):

1) The number of Basis Elements B_i is ∞.

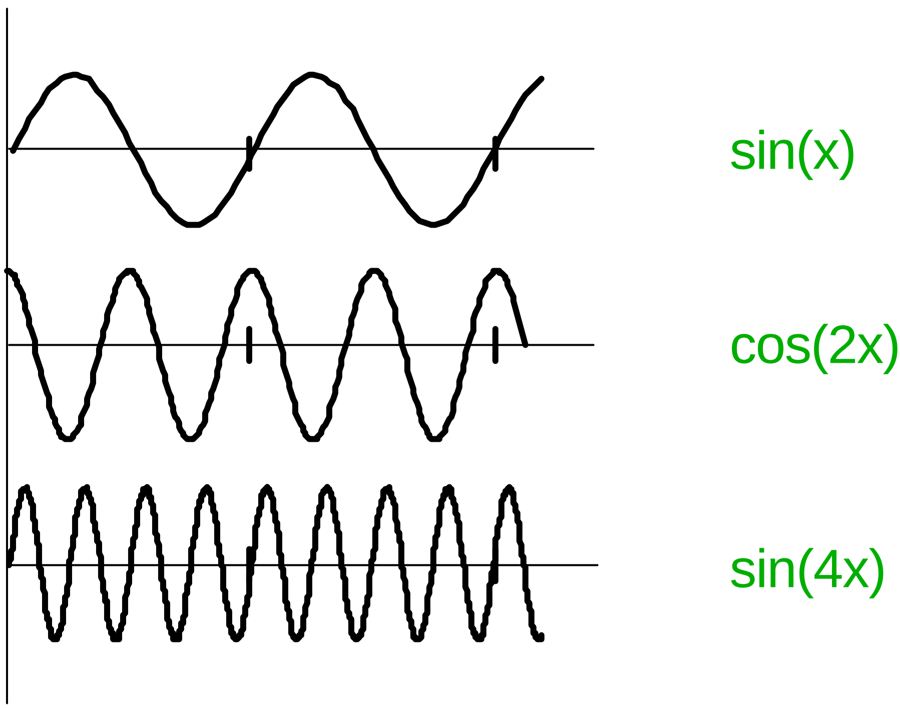
$$f(x) = \int_i a_i B_i(x)\, di$$

2) The dot product:

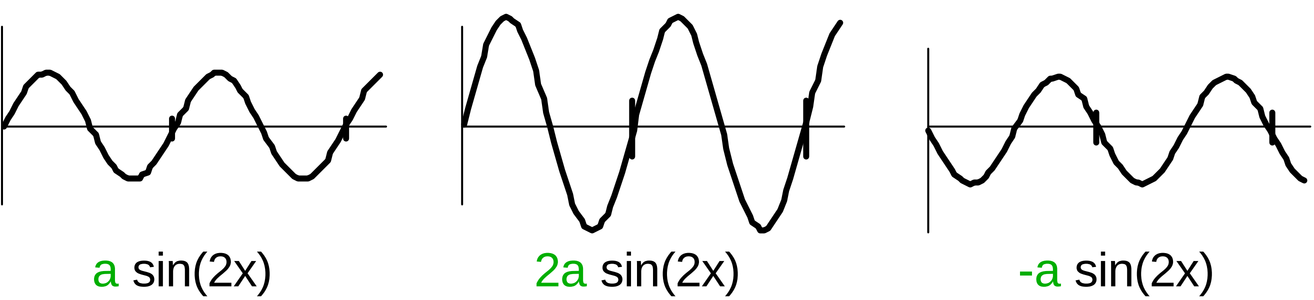
$$\langle f(x), B_i(x) \rangle = \int_x f(x)\, B_i(x)\, dx$$

Fourier Transform

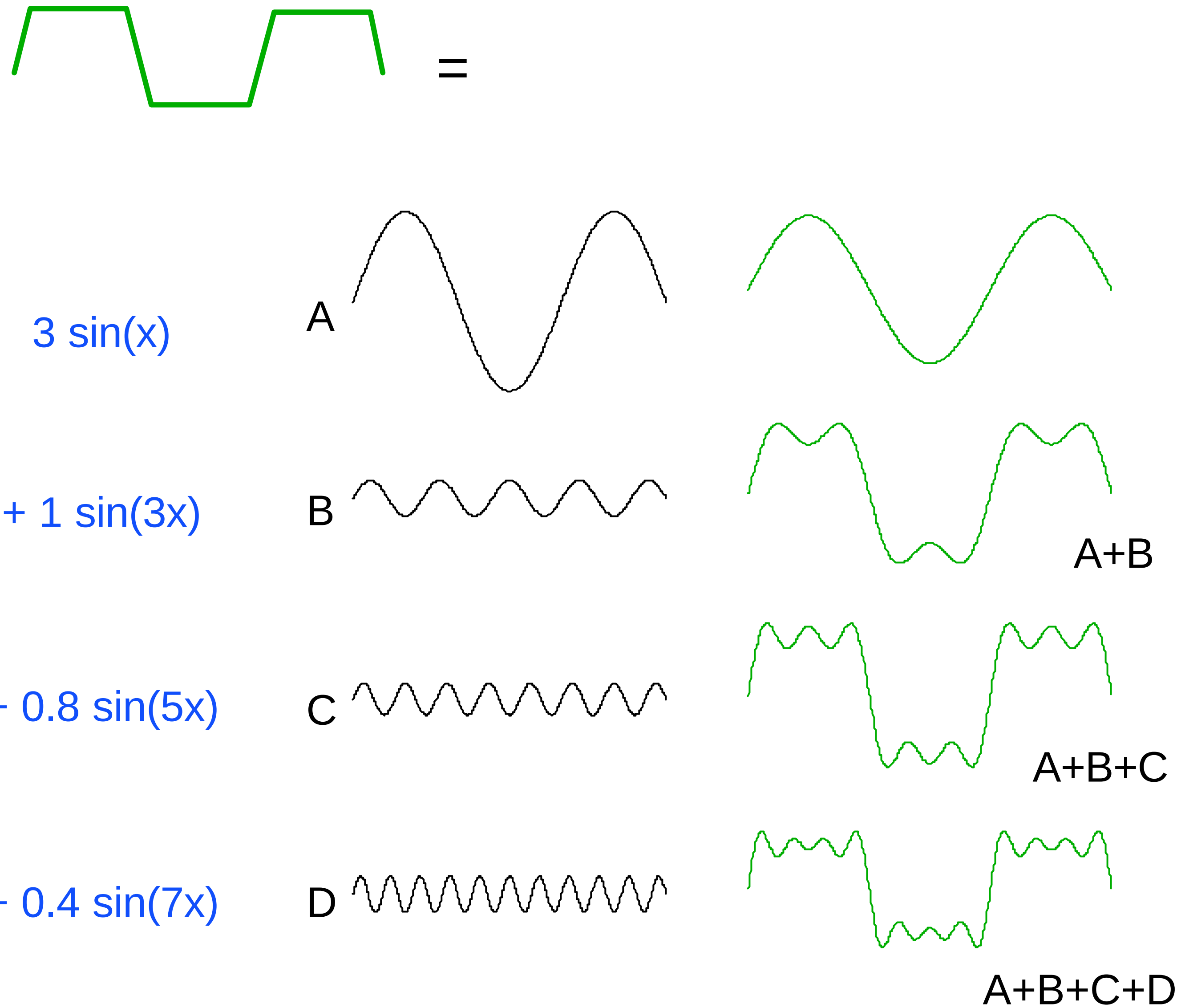
Basis Functions are sines and cosines



The transform coefficients determine the amplitude:



Every function equals a sum of sines and cosines



The Fourier Transform

- The inverse Fourier Transform composes a signal f(x) given $F(w)$

$$f(x) = \int_{w} F(w) e^{i2pwx} \, dw$$

- The Fourier Transform finds the $F(w)$ given the signal f(x):

$$F(w) = \int_{x} f(x) e^{-i2pwx} \, dx$$

- F(ω) is the Fourier transform of f(x):

$$\tilde{F}\{f(x)\} = F(w)$$

- f(x) is the inverse Fourier transform of F(ω):

$$\tilde{F}^{-1}\{F(w)\} = f(x)$$

- f(x) and F(ω) are a Fourier transform pair.

- The Fourier transform F(ω) is a function over the complex numbers:

$$F(w) = R_w e^{iq_w}$$

  - R_ω tells us how much of frequency ω is needed.
  - θ_ω tells us the shift of the Sine wave with frequency ω.

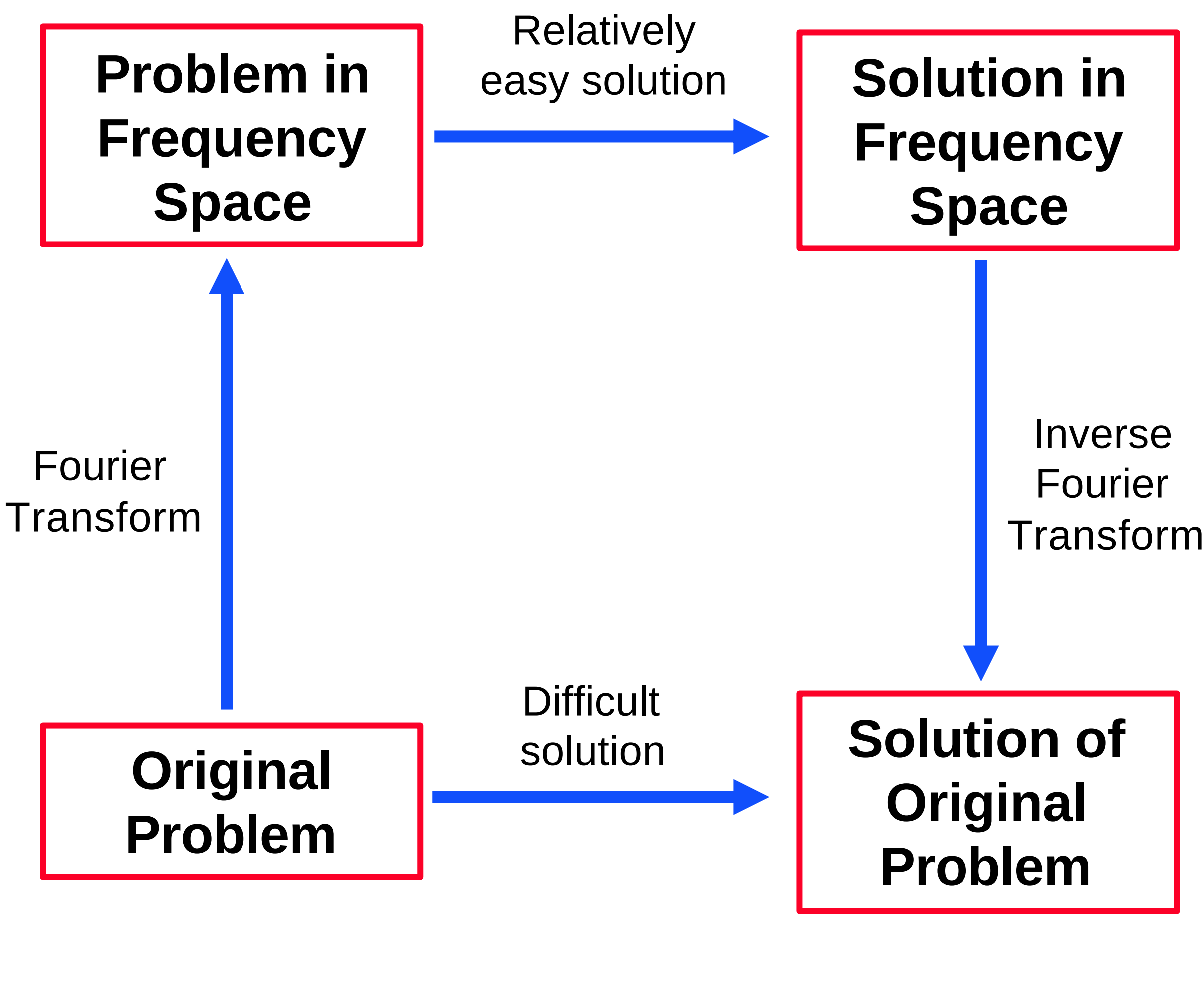
- Alternatively:

$$F(w) = a_w + ib_w$$

  - a_ω tells us how much of cos with frequency ω is needed.
  - b_ω tells us how much of sin with frequency ω is needed.

- R_ω - is the amplitude of $F(\omega)$.
- θ_ω - is the phase of $F(\omega)$.
- $|R_\omega|^2 = F^*(\omega) F(\omega)$ - is the power spectrum of $F(\omega)$.
- If a signal f(x) has a lot of fine details $F(\omega)$ will be high for high ω.
- If the signal f(x) is "smooth" $F(\omega)$ will be low for high ω.

Why do we need representation in the frequency domain?

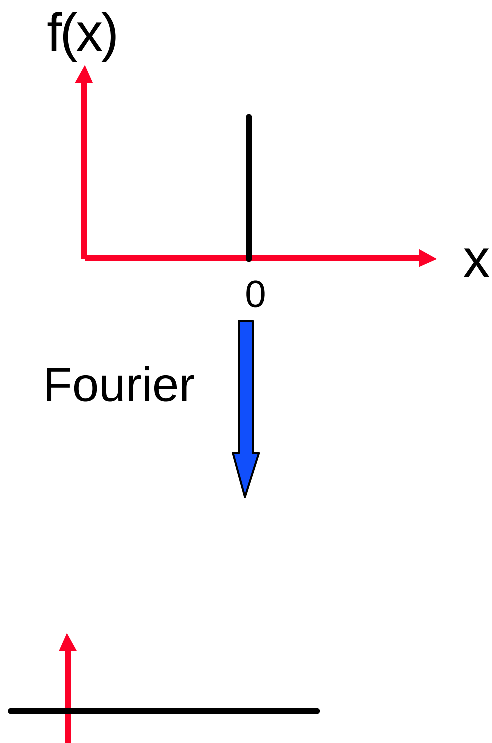


Examples:

The Delta Function:

- Let $$f(x) = d(x)$$

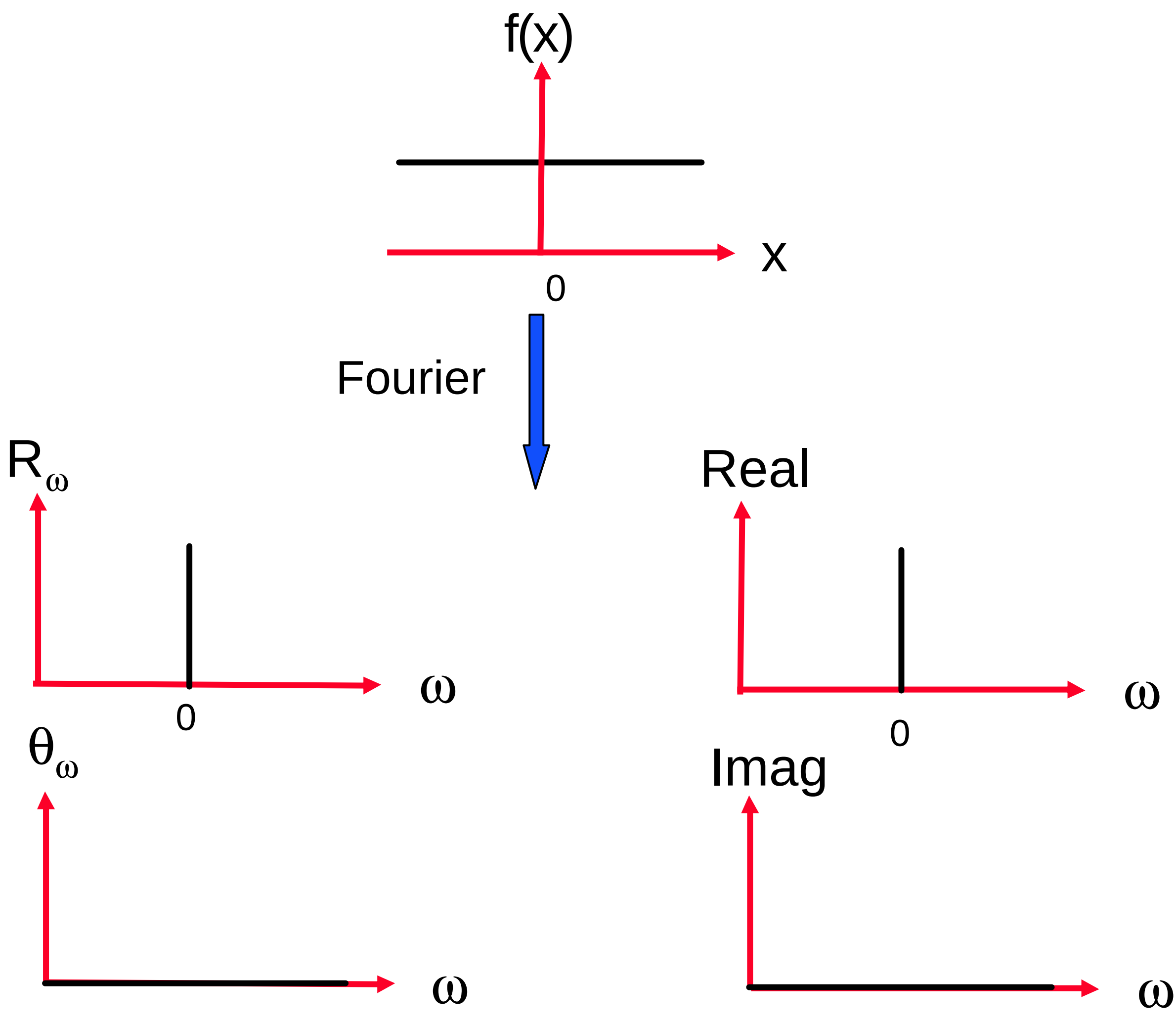
$$F(w) = \int_{-\infty}^{\infty} d(x) \cdot e^{-i2pwx} = 1$$



The Constant Function:

- Let $f(x) = 1$

$$F(w) = \int_{-\infty}^{\infty} e^{-i2pwx} = d(w)$$

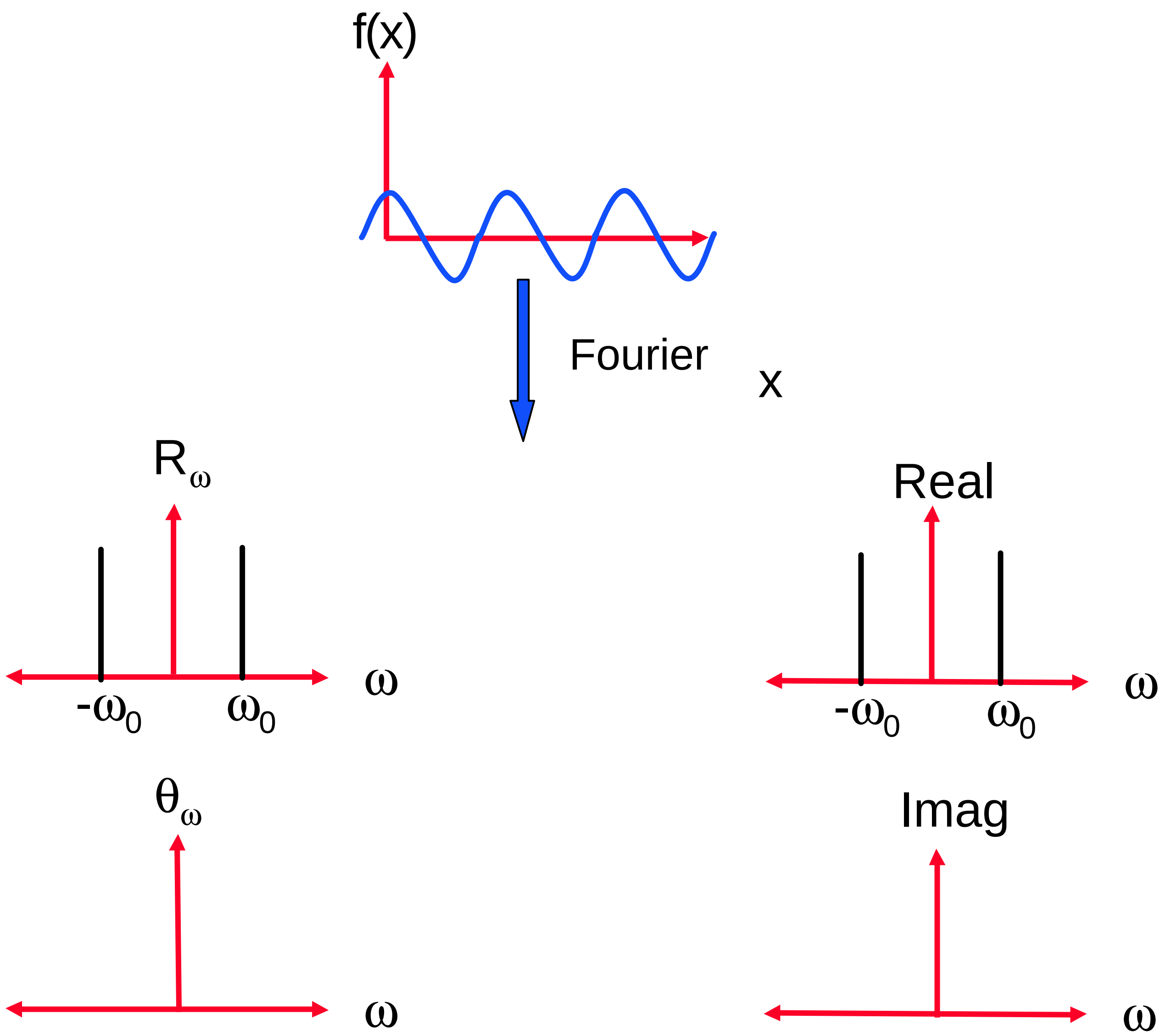


The Cosine wave:

- Let $f(x)=\cos(2pw_0x)$

$$F(w)=\int_{-\infty}^{\infty}\frac{1}{2}\left(e^{i2pw_0x}+e^{-i2pw_0x}\right)\cdot e^{-i2pwx}\,dx=$$

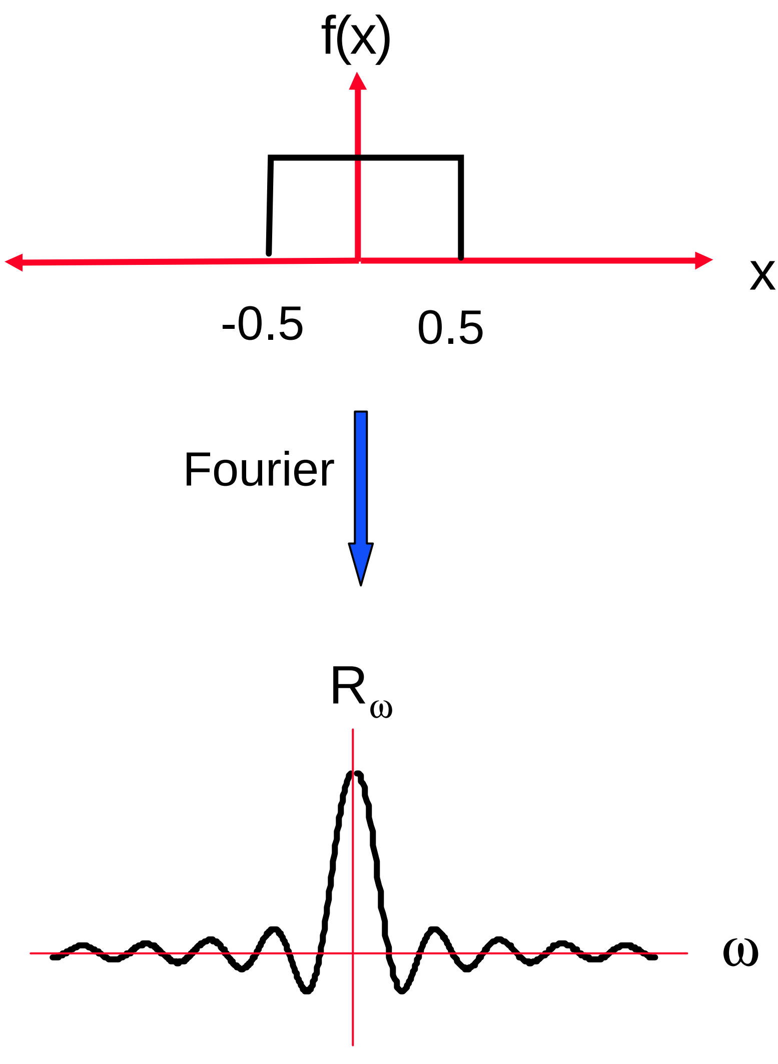
$$=\frac{1}{2}\left[d\left(w-w_0\right)+d\left(w+w_0\right)\right]$$



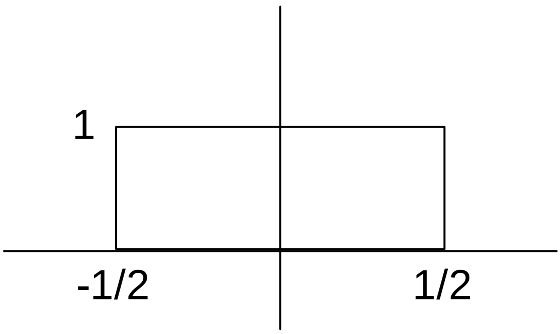
The Window Function (rect):

- Let $\mathrm{rect}_{1/2}(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

$$F(w) = \int_{-0.5}^{0.5} e^{-i2pwx}\,dx = \frac{\sin(pw)}{pw} = \mathrm{sinc}(pw)$$



Proof:



$$f(x) = \text{rect}_{1/2}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx = \int_{-1/2}^{1/2} e^{-2\pi iux} dx$$

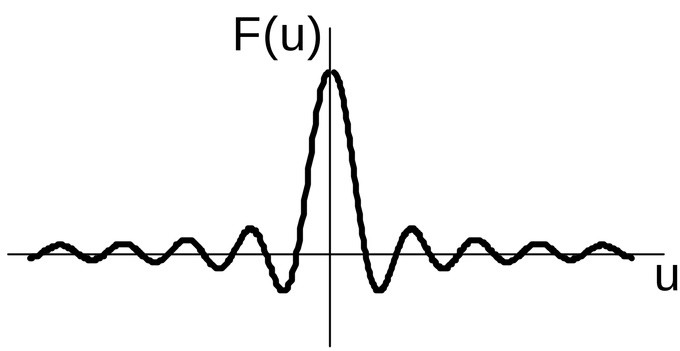
$$= \frac{1}{-2\pi iu} \left[e^{-2\pi iux} \right]_{-1/2}^{1/2}$$

$$= \frac{1}{-2\pi iu} \left[e^{-\pi iu} - e^{\pi iu} \right]$$

$$= \frac{1}{-2\pi iu} \left[\cancel{\cos(\pi u)} - i\sin(\pi u) - \cancel{\cos(\pi u)} - i\sin(\pi u) \right]$$

$$= \frac{\sin(\pi u)}{\pi u}$$ = sinc(u)

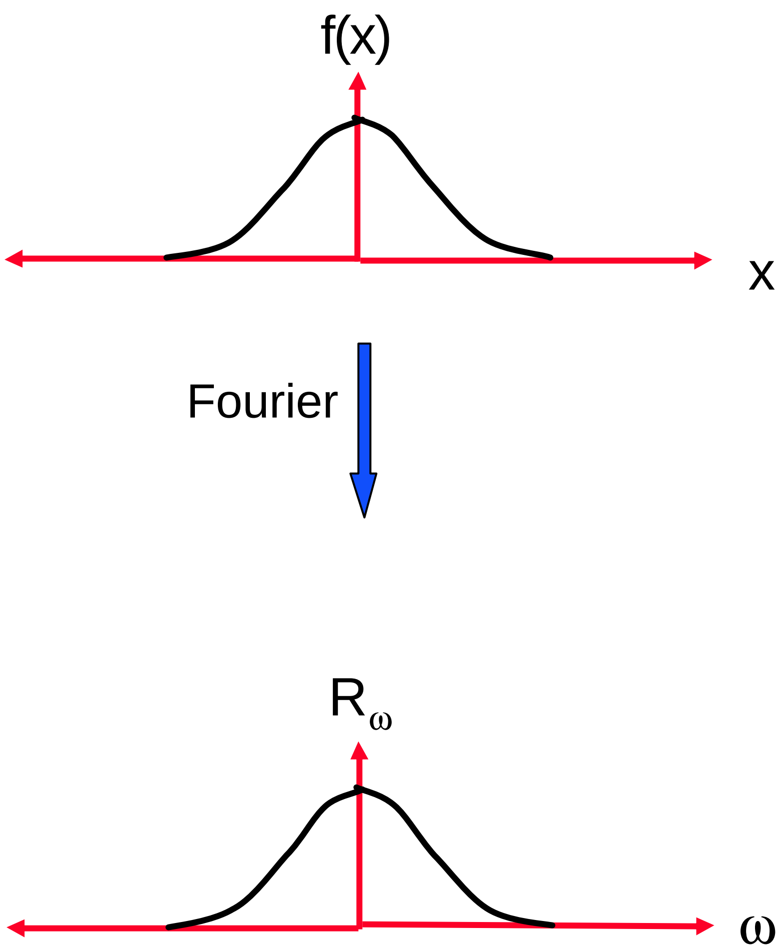
sinc(u)



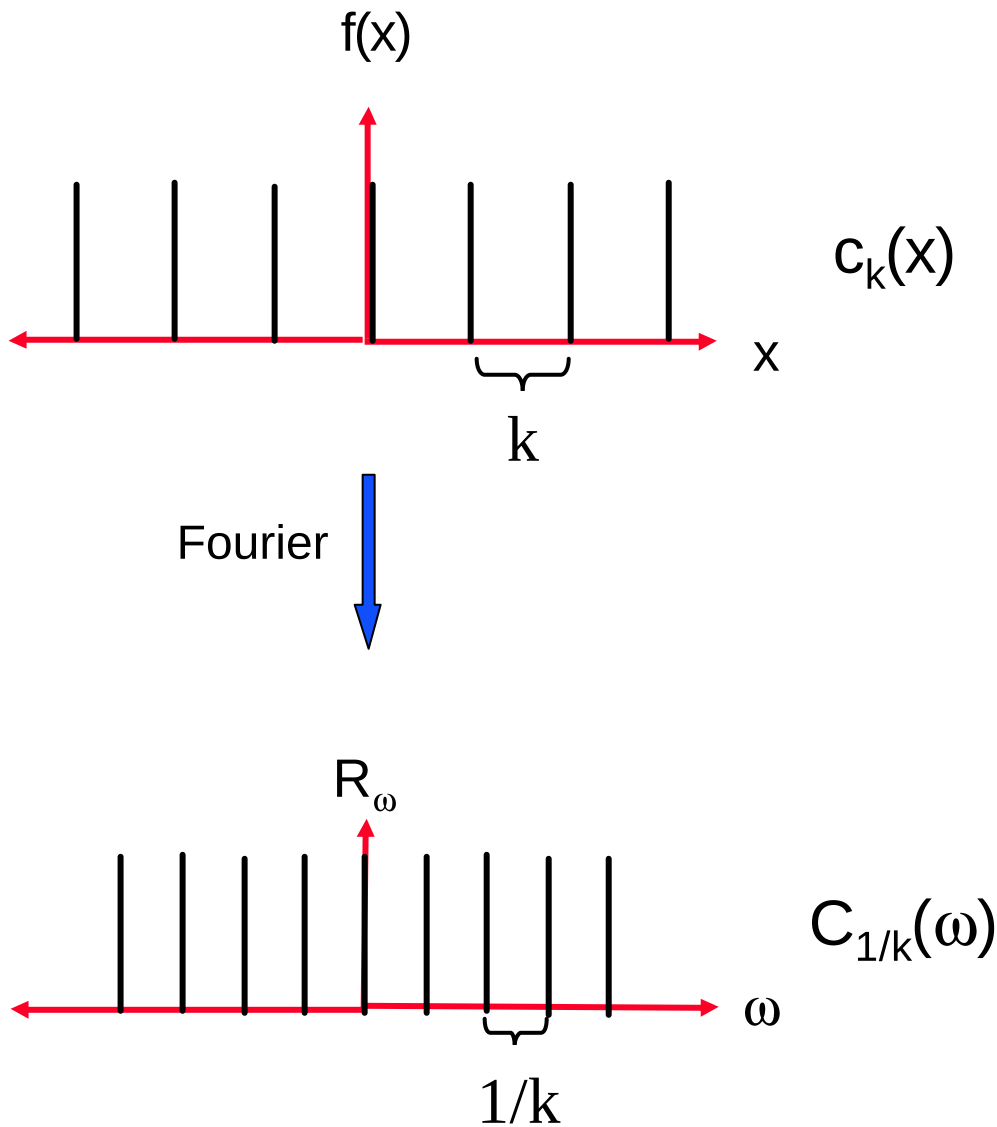
The Gaussian:

- Let $f(x) = e^{-px^2}$

$$F(w) = e^{-pw^2}$$



The bed of nails function:



Fourier Transform - 2D

Given a continuous real function f(x,y),
its Fourier transform F(u,v) is defined as:

$$\tilde{F}\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i(ux+vy)}\,dx\,dy$$

The Inverse Fourier Transform:

$$\tilde{F}^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(ux+vy)}\,du\,dv$$

$$F(u,v) = a(u,v) + ib(u,v) = |F(u,v)|e^{i\phi(u,v)}$$

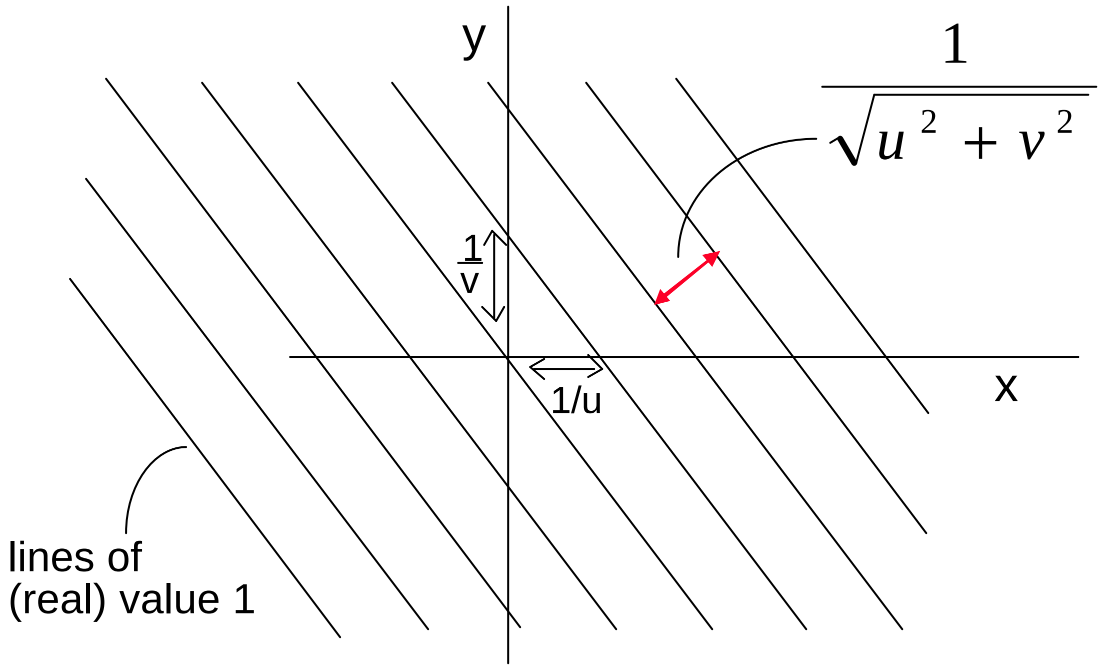
Phase = $\phi(u,v) = tg^{-1}(b(u,v)/a(u,v))$

Spectrum (Amplitude) = $|F(u,v)| = \sqrt{a^2(u,v) + b^2(u,v)}$

Power Spectrum = $|F(u,v)|^2 = a^2(u,v) + b^2(u,v)$

Fourier Wave Functions - 2D

$F(u,v)$ is the coefficient of the sine wave $e^{2\pi i(ux+vy)}$

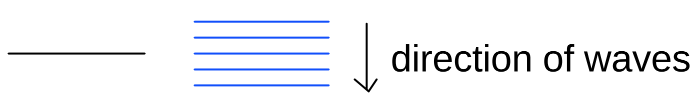


$$e^{2\pi i(ux+vy)} = \cos(2\pi(ux+vy)) + i\sin(2\pi(ux+vy))$$

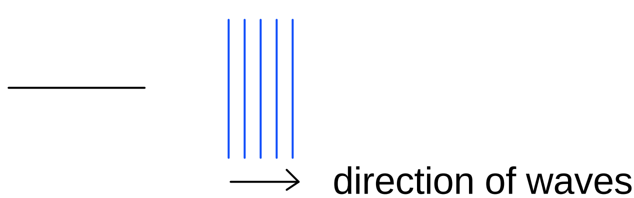
The ratio $\frac{u}{v}$ determines the Direction.

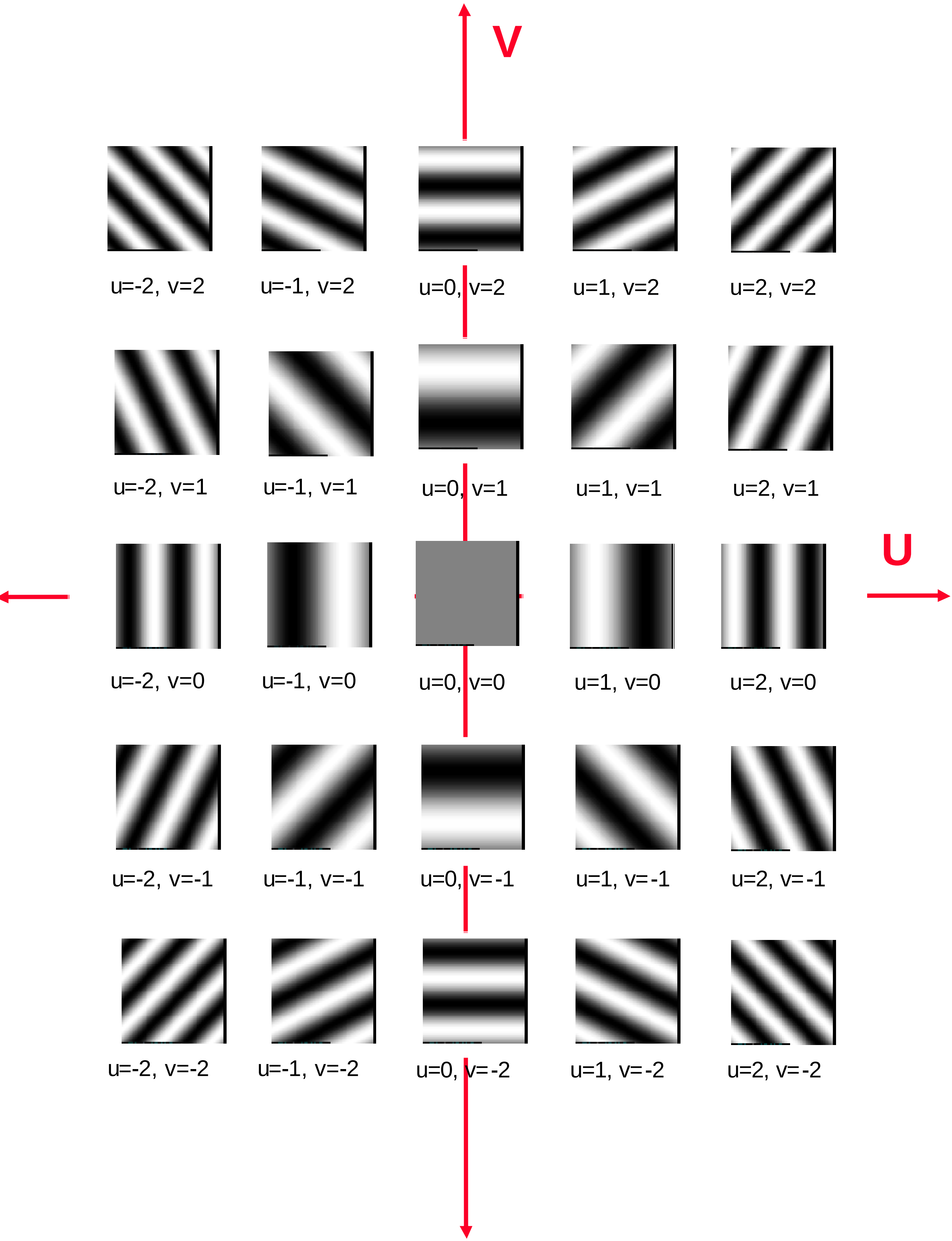
The size of u,v determines the Frequency.

u = 0



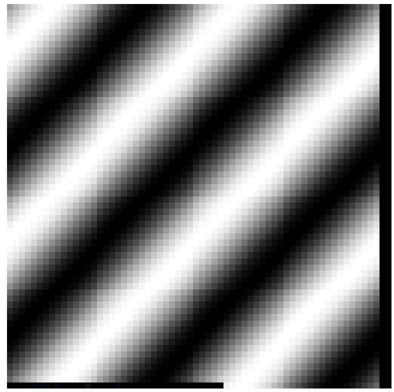
v = 0

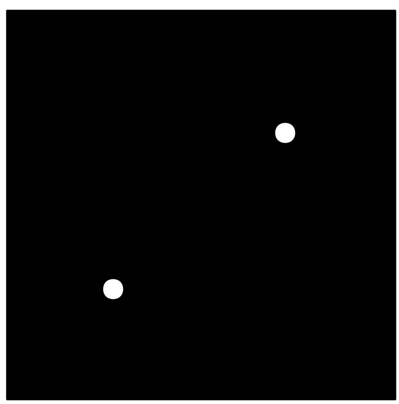



V
U
u=-2, v=2
u=-1, v=2
u=0, v=2
u=1, v=2
u=2, v=2
u=-2, v=1
u=-1, v=1
u=0, v=1
u=1, v=1
u=2, v=1
u=-2, v=0
u=-1, v=0
u=0, v=0
u=1, v=0
u=2, v=0
u=-2, v=-1
u=-1, v=-1
u=0, v=-1
u=1, v=-1
u=2, v=-1
u=-2, v=-2
u=-1, v=-2
u=0, v=-2
u=1, v=-2
u=2, v=-2

Fourier Transform 2D - Example

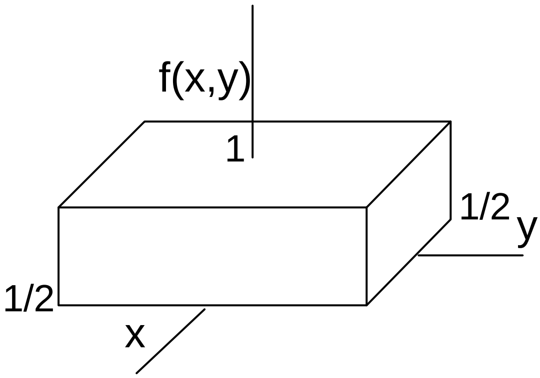
2D Function

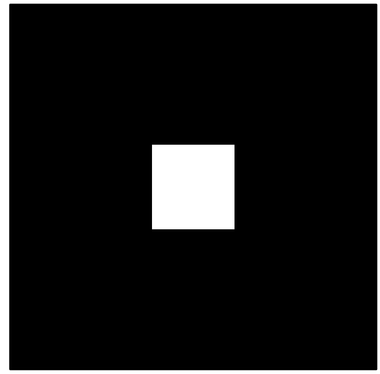




2D Fourier Transform

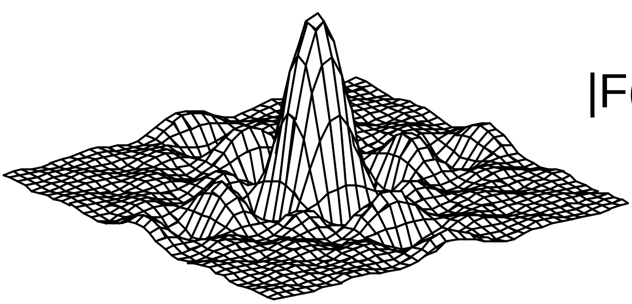
Fourier Transform 2D - Example



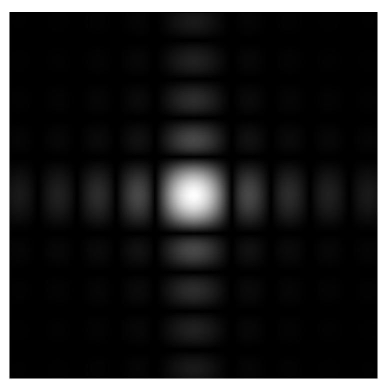


$$f(x,y) = \mathrm{rect}(x,y) = \begin{cases} 1 & |x| \le 1/2,\ |y| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u,v) = \mathrm{sinc}(u) \cdot \mathrm{sinc}(v) = \mathrm{sinc}(u,v)$$



$|F(u,v)|$



Proof of Fourier of Rect = sinc in 2D

$$F(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2pi(ux+vy)} dxdy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-2pi(ux+vy)} dxdy$$

$$= \int_{-1/2}^{1/2} e^{-2p\, iux}\, dx \int_{-1/2}^{1/2} e^{-2p\, ivy}\, dy$$

$$= \frac{\sin(pu)}{pu}\frac{\sin(pv)}{pv} = Sinc(u,v)$$

Fourier Transform Examples

Image Domain | Frequency Domain