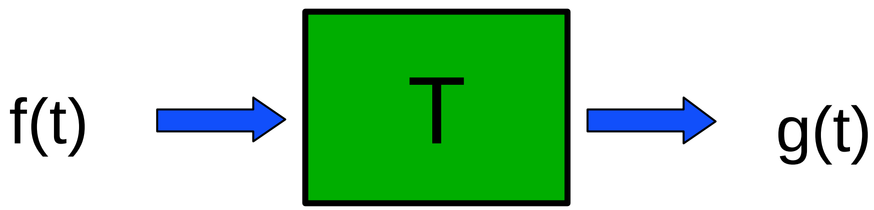
Introduction to Fourier Transform - Linear Systems

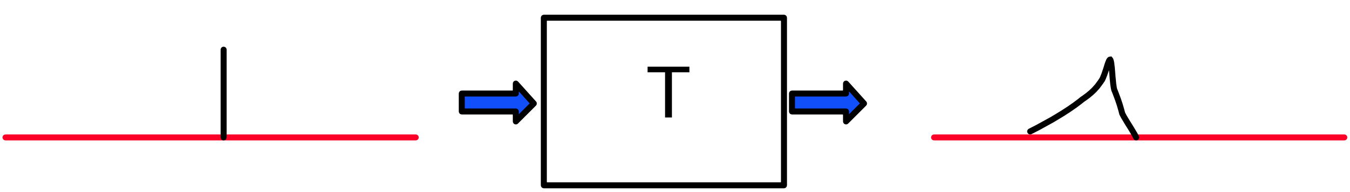
- Linear Systems
  - Definitions & Properties
  - Shift Invariant Linear Systems
  - Linear Systems and Convolutions
  - Linear Systems and sinusoids
  - Complex Numbers and Complex Exponentials
  - Linear Systems - Frequency Response

Linear Systems

- A linear system T gets an input f(t) and produces an output g(t):



$$g(t) = T\{f(t)\}$$

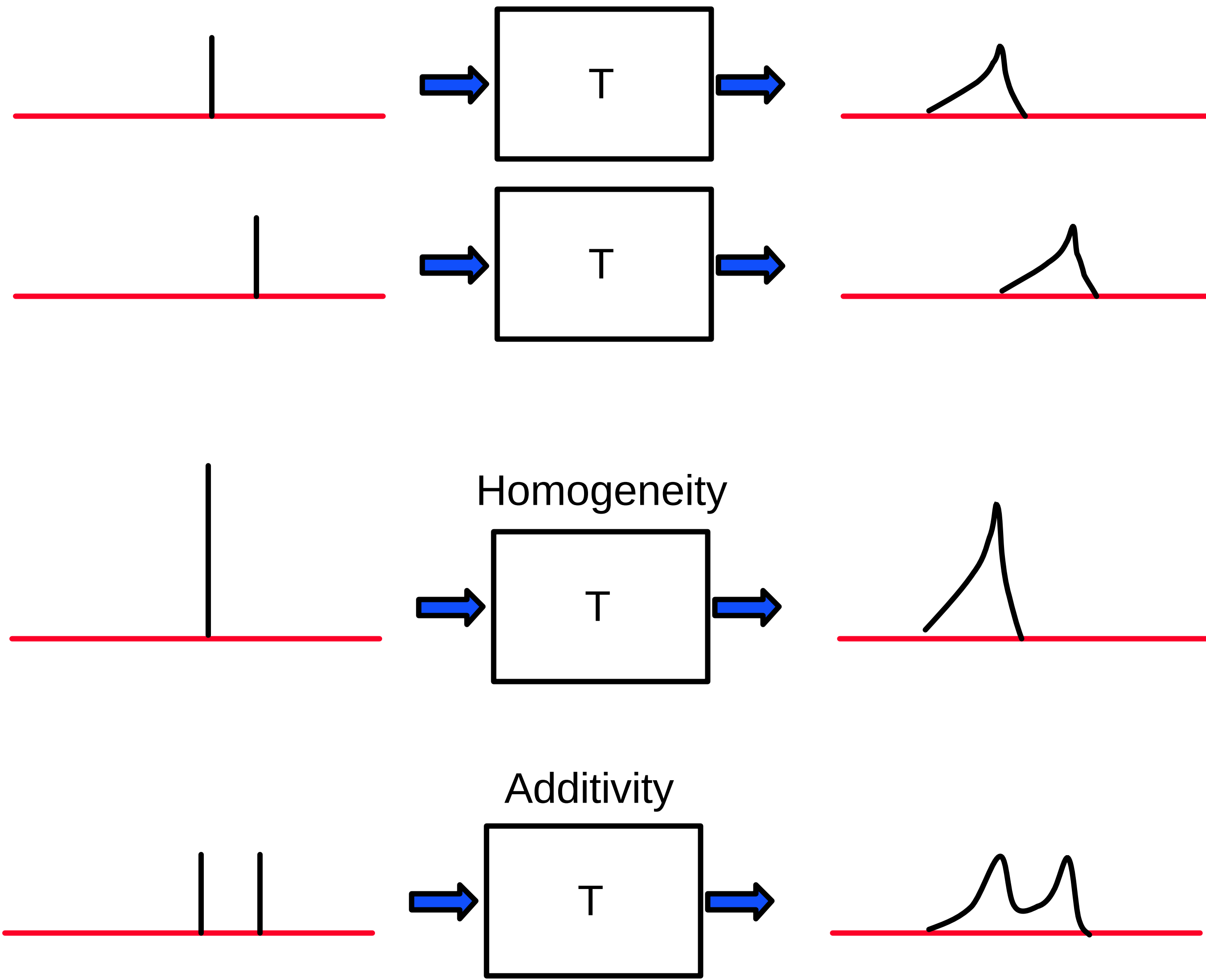


- In the discrete caes:
  - input : f[n] , n = 0,1,2,…
  - output: g[n] , n = 0,1,2,…

$$\mathrm{g}[n] = T[\mathrm{f}(n)]$$

Linear System Properties

- A linear system must satisfy two conditions:
  - Homogeneity: $T\{a\, f[n]\} = aT\{f[n]\}$
  - Additivity: $T\{f_1[n] + f_2[n]\} = T\{f_1[n]\} + T\{f_2[n]\}$



Linear System - Example

- **Contrast change** by grayscale stretching around 0:

$$T\{f(x)\} = af(x)$$

  - Homogeneity:

$$T\{bf(x)\} = abf(x) = baf(x) = bT\{f(x)\}$$

  - Additivity:

$$\begin{aligned} T\{f_1(x)+f_2(x)\} &= a(f_1(x)+f_2(x)) \\ &= af_1(x)+af_2(x) \\ &= T\{f_1(x)\}+ T\{f_2(x)\} \end{aligned}$$

Linear System - Example

- **Convolution**:

$$T\{f(x)\} = f*a$$

  – Homogeneity:

$$T\{bf(x)\} = (bf)*a = b(f*a) = bT\{f(x)\}$$

  – Additivity:

$$\begin{aligned} T\{f_1(x)+f_2(x)\} &= (f_1+f_2)*a \\ &= f_1*a+f_2*a \\ &= T\{f_1(x)\}+ T\{f_2(x)\} \end{aligned}$$

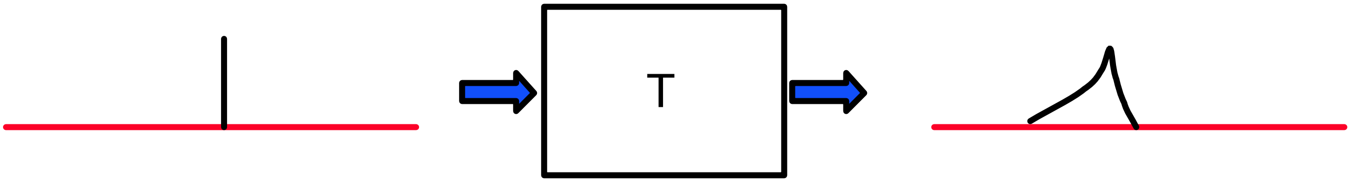
Shift-Invariant Linear System

- Assume T is a linear system satisfying

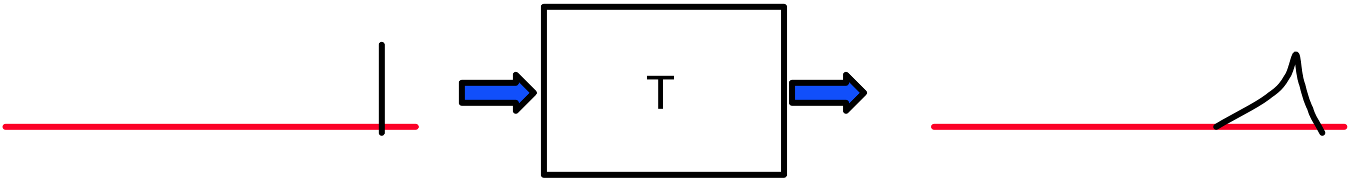
$$g(t) = T\{f(t)\}$$

- T is a shift-invariant linear system iff:

$$g(t-t_0) = T\{f(t-t_0)\}$$



Shift Invariant



Shift-Invariant Linear System - Example

- **Contrast change** by grayscale stretching around 0:

$$T\{f(x)\} = af(x) = g(x)$$

  – Shift Invariant:

$$T\{f(x-x_0)\} = af(x-x_0) = g(x-x_0)$$

- **Convolution**:

$$T\{f(x)\} = f(x)*a = g(x)$$

  – Shift Invariant:

$$T\{f(x-x_0)\} = f(x-x_0)*a$$

$$= \sum_i f(i-x_0)a(x-i) = \sum_j f(j)a(x-j-x_0)$$

$$= g(x-x_0)$$

Matrix Multiplication as a Linear System

- Assume f is an input vector and T is a matrix multiplying f:

$$g = Tf$$

- g is an output vector.
- Claim: A matrix multiplication is a linear system:
  - Homogeneity $T(af) = aTf$
  - Additivity $T(f_1 + f_2) = Tf_1 + Tf_2$
- Note that a matrix multiplication is not necessarily shift-invariant.

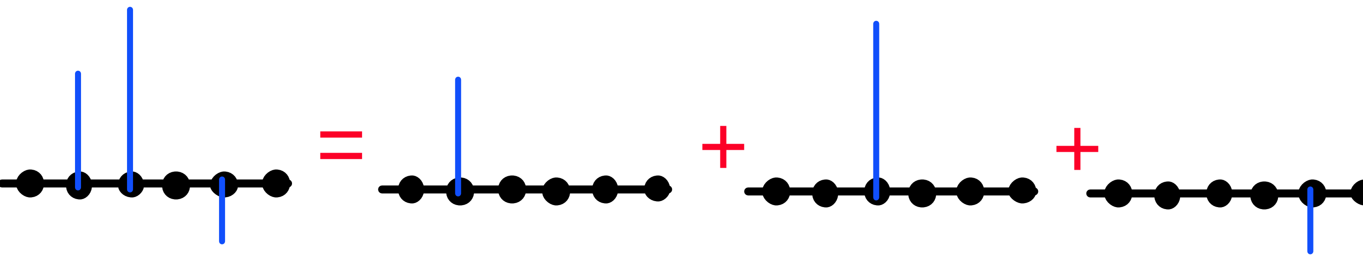
Impulse Sequence

- An impulse signal is defined as follows:

$$d[n-k]=\begin{cases}0 & \text{where} \quad n \neq k \\ 1 & \text{where} \quad n = k\end{cases}$$

- Any signal can be represented as a linear sum of scales and shifted impulses:

$$f[n]=\sum_{j=-\infty}^{\infty} f[j]\,d[n-j]$$



Shift-Invariant Linear System is a Convolution

Proof:

- f[n] input sequence
- g[n] output sequence
- h[n] the system impulse response:

h[n]=T{δ[n]}

$$g[n]=T\{f[n]\}=T\left\{\sum_{j=-\infty}^{\infty} f[j]d[n-j]\right\}$$

$$=\sum_{j=-\infty}^{\infty} f[j]T\{d[n-j]\} \quad (from\ linearity)$$

$$=\sum_{j=-\infty}^{\infty} f[j]h[n-j] \quad (from\ shift-inariancce)$$

$$= \quad f*h$$

The output is a sum of scaled and shifted copies of impulse responses.

Convolution as a Matrix Multiplication

- The convolution (wrap around):

$$[1 \;\; 2 \;\; 0 \;\; 0 \;\; -1 \;\; -2] * [3 \;\; 2 \;\; 1] = [6 \;\; 5 \;\; 2 \;\; -3 \;\; -8 \;\; -2]$$

can be represented as a matrix multiplication:

Circulant Matrix →

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ -3 \\ -8 \\ -2 \end{bmatrix}$$

- The matrix rows are flipped and shifted copies of the impulse response.
- The matrix columns are shifted copies of the impulse response.

Convolution Properties

- Commutative:

$$T_1 * T_2 * f = T_2 * T_1 * f$$

  - Only shift-invariant systems are commutative.
  - Only circulant matrices are commutative.

- Associative:

$$(T_1 * T_2) * f = T_1 * (T_2 * f)$$

  - Any linear system is associative.

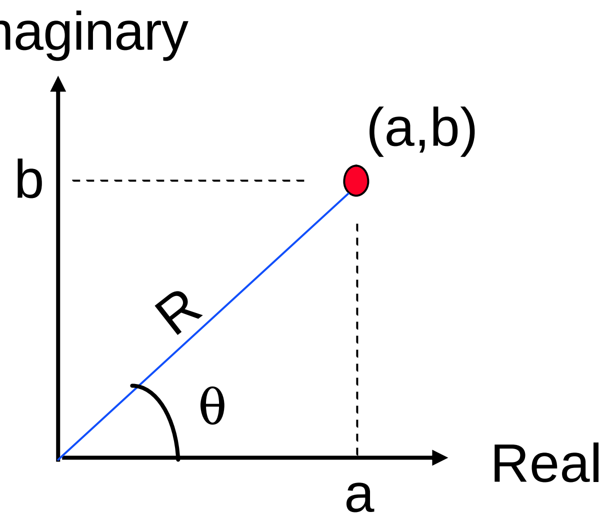
- Distributive:

$$(T_1 + T_2) * f = T_1 * f + T_2 * f$$

$$\text{and } \; T * (f_1 + f_2) = T * f_1 + T * f_2$$

  - Any linear system is distributive.

Complex Numbers



The Complex Plane

- Two kind of representations for a point (a,b) in the complex plane
  - The Cartesian representation:

$$Z = a + bi \qquad where \quad i^2 = -1$$

  - The Polar representation:

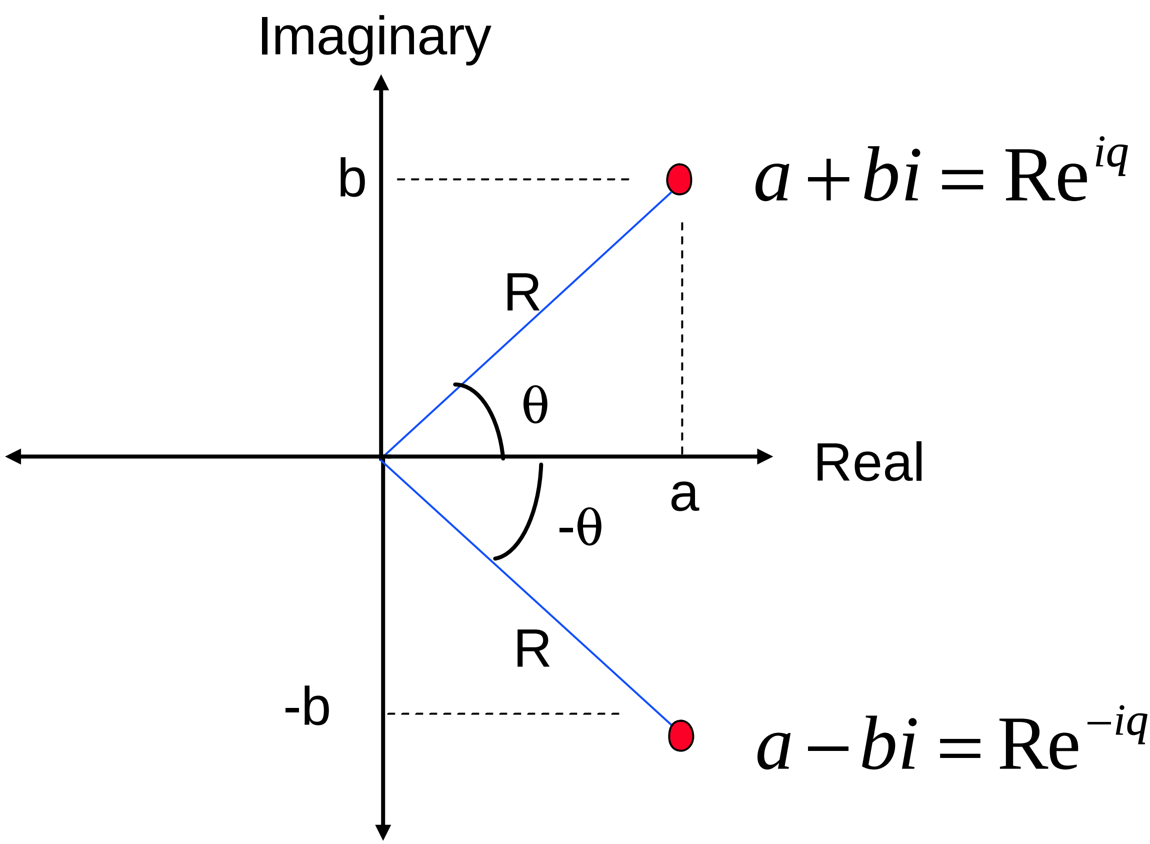
$$Z = \mathrm{R}e^{iq}$$ (Complex exponential)

- Conversions:
  - Polar to Cartesian: $\mathrm{R}e^{iq} = R\cos(q) + iR\sin(q)$
  - Cartesian to Polar $a + bi = \sqrt{a^2 + b^2}\; e^{i\tan^{-1}(b/a)}$

- Conjugate of Z is Z^*:

  – Cartesian rep. $(a + ib)^* = a - ib$

  – Polar rep. $\left(Re^{iq}\right)^* = Re^{-iq}$



Algebraic operations:

- addition/subtraction:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

- multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

$$Ae^{ia}\ Be^{iß}=ABe^{i(a+ß)}$$

- Norm:

$$\|a+ib\|^2=(a+ib)^*(a+ib)=a^2+b^2$$

$$\left\|Re^{iq}\right\|^2=\left(Re^{iq}\right)^* Re^{iq}=Re^{-iq}\ Re^{iq}=R^2$$

The (Co-) Sinusoid

- The (Co-)Sinusoid as complex exponential:

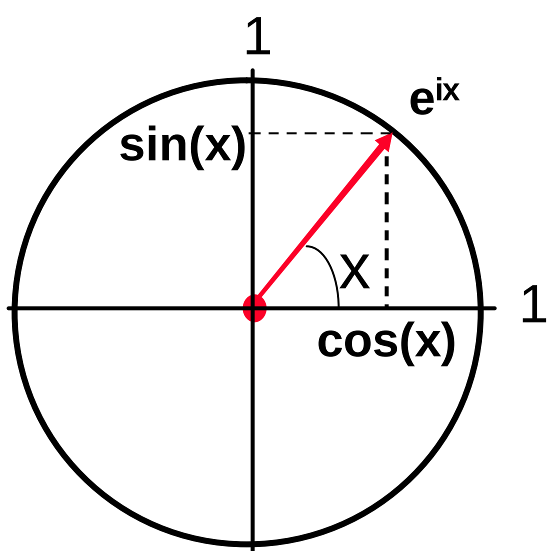
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

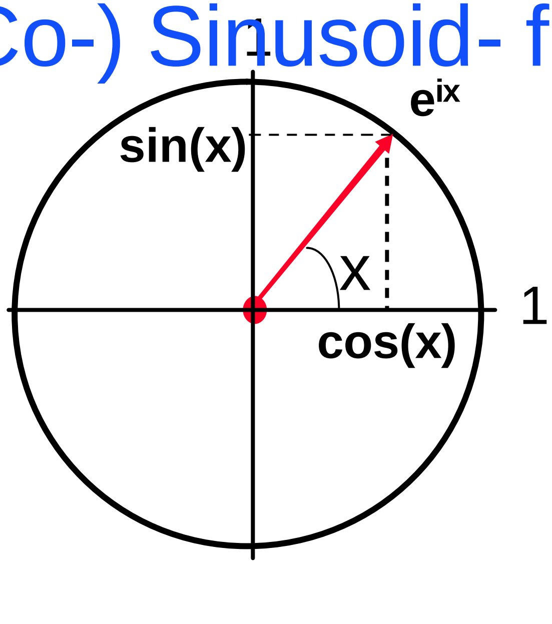
Or

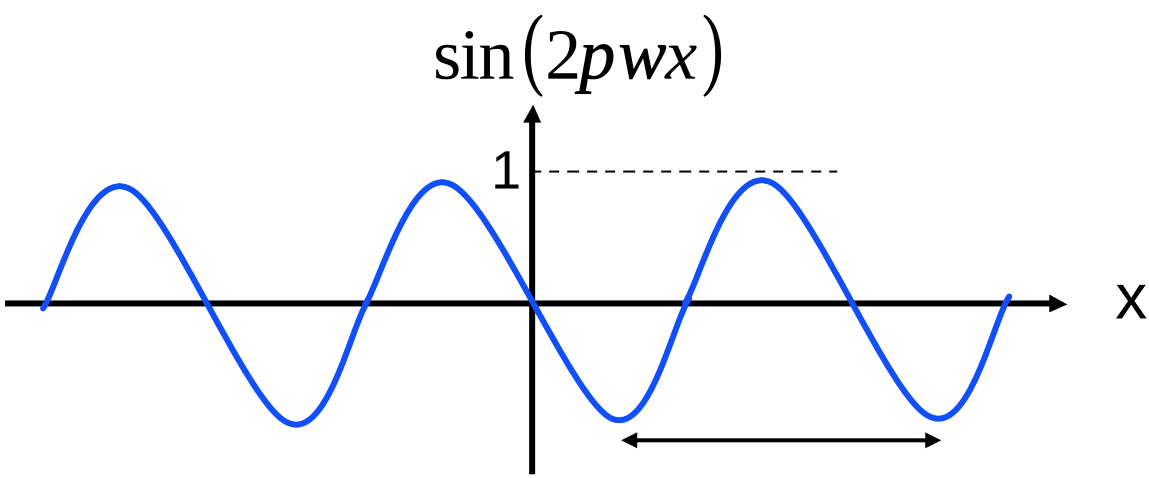
$$\cos(x) = \mathrm{Real}\left(e^{ix}\right)$$

$$\sin(x) = \mathrm{Imag}\left(e^{ix}\right)$$



The (Co-) Sinusoid- function



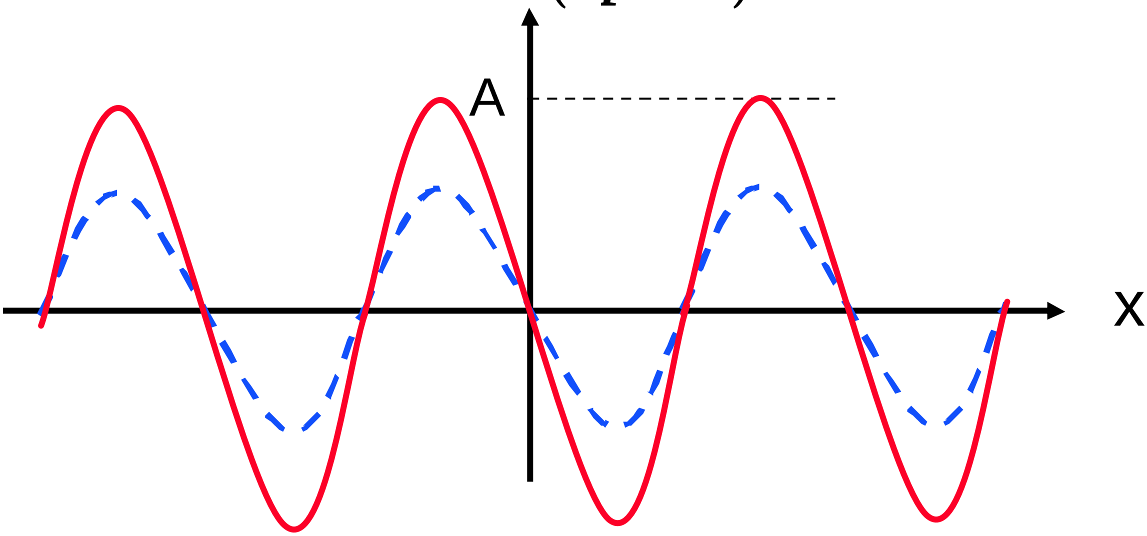


– The wavelength of $\sin(2pwx)$ is $\frac{1}{w}$.

– The frequency is w.

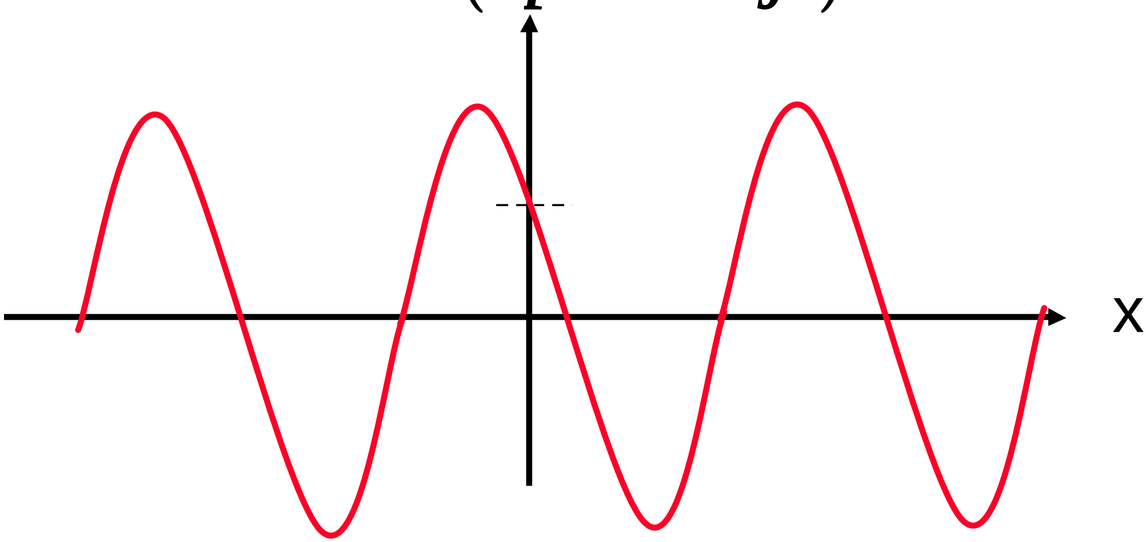
– Changing Amplitude:

$$A\sin\left(2pwx\right)$$



– Changing Phase:

$$A\sin\left(2pwx+j\right)$$



Scaling and shifting can be represented as a multiplication with Ae^{ij}

$$A\sin\left(2pwx+j\right)=\mathrm{Imag}(Ae^{ij}\;e^{i2pwx})$$

Frequency Analysis

- If a function f(x) can be expressed as a linear sum of scaled and shifted sinusoids:

$$f(x) = \sum_{w} F(w) e^{i2pwx}$$

it is possible to predict the system response to f(x):

$$g(x) = T\{f(x)\} = \sum_{w} H(w) F(w) e^{i2pwx}$$

- The Fourier Transform:

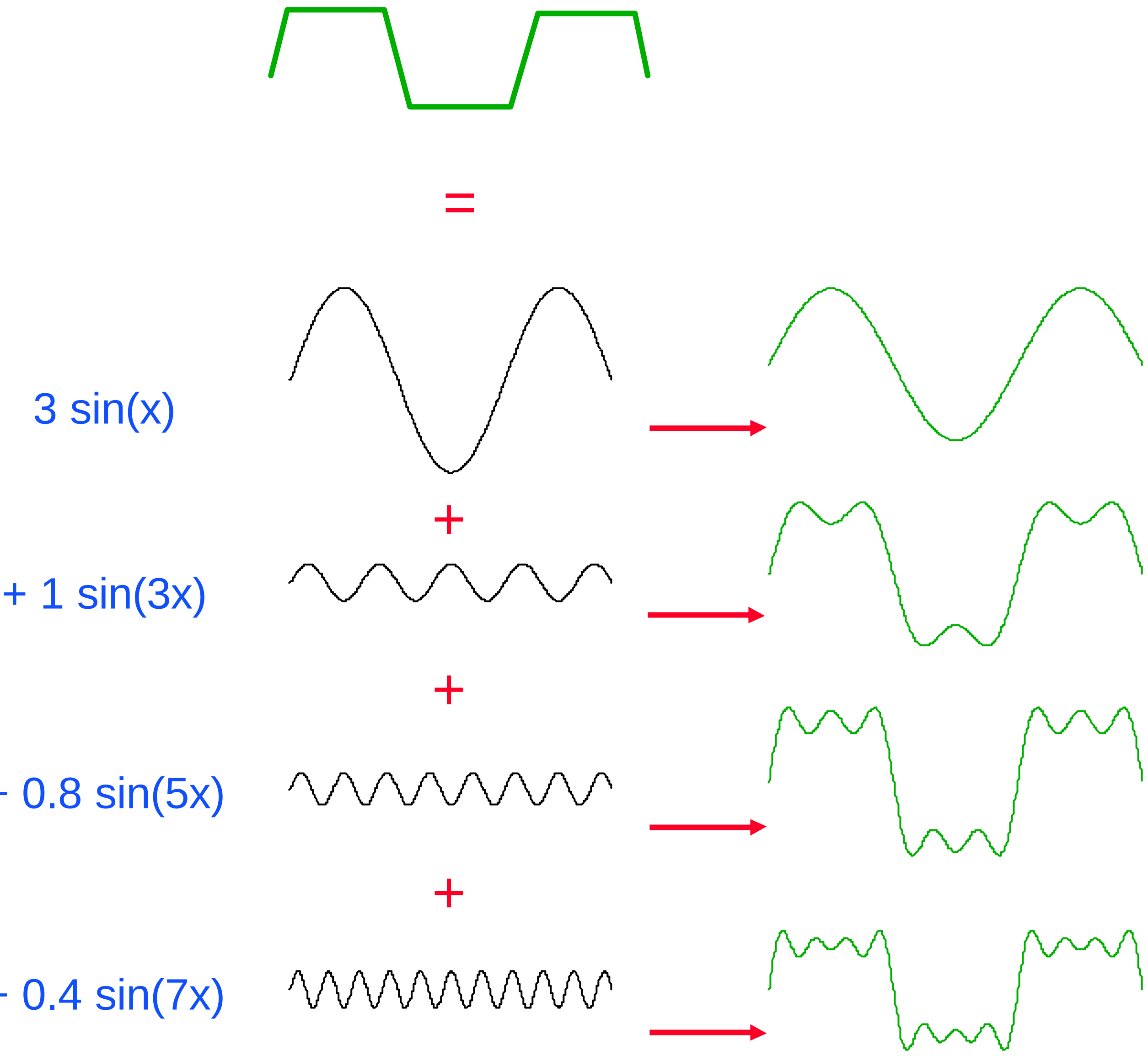
It is possible to express any signal as a sum of shifted and scaled sinusoids at different frequencies.

$$f(x) = \sum F(w) e^{i2pwx}$$

Or

$$f(x) = \int_{w} F(w) e^{i2pwx}\, dw$$

Every function equals a sum of scaled and shifted Sines



Linear System Logic

Input Signal

Frequency Method

1. Express as sum of scaled and shifted sinusoids
2. Calculate the response to each sinusoid
3. Sum the sinusoidal responses to determine the output

$$G(w) = F(w)\,H(w)$$

space/time Method

1. Express as sum of scaled and shifted impulses
2. Calculate the response to each impulse
3. Sum the impulse responses to determine the output

$$g(x) = f(x) * h(x)$$