# New Approach to the Arc Length Parameterization Problem *

Joseph (Yossi) Gil‡
The Faculty of Computer Science
The Technion—Israel Institute of Technology
Technion City
Haifa 32000, ISRAEL

Daniel Keren
Department of Mathematics and Computer Science
Haifa University
Haifa 31905, ISRAEL

May 25, 1997

## Abstract

Arc length parameterization can be thought of as the most natural among all possible parameterizations of a given curve. Beyond having several nice mathematical properties, this parameterization is useful for computer graphics applications: drawing a curve given in this form and computing its length, are particularly easy. Unfortunately, for the curves used mostly in computer graphics, namely cubic splines, the arc length parameterization cannot, in general, be expressed as any elementary function. As a result, practitioners are forced into employing approximate numerical methods which are in many cases complex, computationally intensive, and susceptible to error accumulation due to their iterative nature.

This paper explores the following new direction to this problem. Instead of the traditional classes of curves, such as polynomials and rational functions, we promote the usage of a new class of curves which are all given in an *explicit* arc length parameterization form. On a par with polynomials, it is possible to select a subclass of curves which has any desired number of degrees of freedom. Our results show that several important settings of the general interpolation problem can be solved using curves from this class.

## 1 Introduction

### 1.1 Curves and Arc Length Parameterization

The most common way of describing curves in computer graphics, computer aided design, computer vision, and other areas of computer science which deal with the geometric properties of objects, is the so-called *parametric representation*. A plane curve, for instance, is described as a pair of functions of one variable, $\langle x(t), y(t)\rangle$, and a range $[t_0, t_1]$. As $t$ ranges between $t_0$ and $t_1$, the point $\langle x(t), y(t)\rangle$ traverses a curve in the plane. The curves most commonly used in graphic applications are *cubics* or *cubic splines* in which both $x(t)$ and $y(t)$ are polynomials of degree 3 of the parameter. Of a lesser usage, but also very important, are the general *polynomial curves*, in which these two functions are polynomials of an arbitrary degree and the *rational curves*, in which they can be any rational function.

There are many different parameterizations for any given geometrical curve. Let $s(t')$ be a monotonically increasing function, and let $t'_0$ and $t'_1$ be such that, $s(t'_0) = t_0$ and $s(t'_1) = t_1$. Then,

$$\langle x(t), y(t)\rangle \qquad t \in [t_0, t_1]$$

and

$$\langle x \circ s(t'), y \circ s(t')\rangle \qquad t' \in [t'_0, t'_1]$$

are *equivalent* parameterizations [1] of the same curve.

An *intrinsic* parameterization, which depends only on the geometry of the curve, is given by substituting the *curve length*, measured from its beginning, for the parameter. A simple example for a curve already given in the so called *arc length parameterization* form [11] is the straight line

$$c + \langle \cos\theta, \sin\theta\rangle\, t \ , \qquad (1)$$

*Research supported by a generous grant from the Israel Science Foundation, founded by *The Israel Academy of Sciences and Humanities*.

†Contact author: yogi@CS.Technion.AC.il

‡Research supported by the generous support of the V.P. fund for the promotion of research at the Technion—G.S. Elkin research fund.

where c is some fixed vector and $\theta$ is an arbitrary scalar.

Arc length parameterization of a curve can be constructed from any other differentiable parameterization by the following conceptually simple two-step process:

1. **Rectification** The cumulative arc length of the curve $\langle x(t), y(t)\rangle$, measured from the point $t = a$, is computed by

$$\ell(t) = \int_a^t \sqrt{x'^2(\tau) + y'^2(\tau)}\, d\tau \ . \tag{2}$$

   We call $\ell(t)$ the *arc length function* of the curve.

2. **Inversion** To produce the arc length parameterization, the function $\ell^{-1}$, a functional inverse of arc length function, must be determined. (This function is well defined and monotonically increasing except for very degenerate cases in which the curve "stalls", i.e., there exists an interval of length greater than zero in which both functions $x$ and $y$ are constant. Hereafter, we limit the discussion to *regular* curves in which the derivative vector never vanishes.) The arc length parameterization is now given by the functions $x \circ \ell^{-1}$ and $y \circ \ell^{-1}$ and by the interval $[0, \ell(b)]$.

For example, applying the above process to the rational curve,

$$c + k\left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle \qquad t \in [-1, 1] \tag{3}$$

using only elementary mathematical tools, reveals that it is actually half a circle

$$c + k\langle \sin t, -\cos t\rangle \qquad t \in [0, \pi] \ . \tag{4}$$

## 1.2 Mathematical Properties of Arc Length Parameterization

A necessary and sufficient condition for a curve to be in arc length parameterization form is that for all $t$,

$$\ell(t) = t \ . \tag{5}$$

By substituting (2) for $\ell(t)$ and taking the derivative with respect to $t$, we get another necessary and sufficient condition for arc length parameterization

$$x'^2(t) + y'^2(t) = 1 \ . \tag{6}$$

The *hodograph* [15] of the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is its parametric derivative $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$. Equation (6) is equivalent to the condition that the hodograph lies on the unit circle centered at the origin

$$|\mathbf{r}'(t)| = 1 \ . \tag{7}$$

Thus, we have that the arc length parameterization is nothing but the parameterization describing a traversal at unit-speed of the geometric curve. Indeed, the arc length parameterization was called in the literature ([5] the *unit-speed representation* of a curve.

Some other well known properties of this representation are, that at any point on the curve, the direction of the tangent to the curve is the same as that of $\mathbf{r}'(t)$. Further, the second derivative $\mathbf{r}''(t)$ is perpendicular to $\mathbf{r}'(t)$ and points to the center of the tangent circle while the radius of that circle is $1/|\mathbf{r}''(t)|$. The curvature of the curve at any of its points is given by

$$\kappa(t) = |\mathbf{r}''(t)| \ . \tag{8}$$

## 1.3 Application of Arc Length Parameterization

Arc length parameterization is also lucrative from a practical point of view. In the following, we have enumerated some of the reasons why it is considered advantageous.

1. When tracing or rendering a curve, it is important to move along it at a constant rate. Otherwise, the points which define the curve will be spread along it at non-regular intervals. Suppose that $\ell'(t)$ is not a constant, and and that $t$ is simply incremented by a fixed constant in each rendering step. Then, the curve will be dense when length increases slowly as a function of $t$, and sparse when it increases quickly. This, of course, has an undesirable effect on the appearance of the curve.

2. Consider the problem of computing the intersection of two curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$. Usually, it is impossible to solve this problem by equating $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$, and analytically solving for $t$ and $u$, as there is no explicit solution for the resulting equations. Instead, numerical methods are used. These usually consist of updating the two parameters $t$ and $u$ as well as the two curves until the distance between $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$ is smaller than some pre-set threshold. It is important to maintain a more or less fixed rate at which the two curves gain length as $t$ and $u$ are updated. If this rate is too fast, then the intersection point might be skipped; if it is too slow, adverse affects on the computation time may be observed.

3. In animation, one of the fastest growing disciplines of computer graphics, it is essential that the speed at which objects are moved along a path can be determined by the animator and not by the particular mathematical parameterization used for describing this path. Further, if an animated object such as a rod or a string changes its shape, it is in many cases vital to preserve its size. This means that it is necessary to gain control of the length of the contour curves. Such control is readily available in arc length parameterization with which the length of the curve segment the two parameter points $t_1$ and $t_2$ is simply $t_2 - t_1$.

4. The generation of ornamental patterns is done by the production of *offset curves* [10, 4, 3] (also called *parallel curves* [13, pp. 42–43]) to a curve given by a designer. The offset curve of a curve $\mathbf{r}(t)$ is $\mathbf{r}(t)+d\mathbf{n}(t)$ where $\mathbf{n}(t)$ is the unit normal to $\mathbf{r}(t)$ and $d$ is a scalar determining the distance of the offset. Offset curves are also a valuable tool in robotics and in cartography. As explained above, offset curves take a particularly simple form, $\mathbf{r}(t) + d\mathbf{r}'(t)$, for curves given in arc length parameterization.

Despite the practical benefits they carry and the mathematically spruce features they exhibit, arc length parameterized curves are rarely used in practice. Our bibliographic search failed to yield any indication of such usage except, of course, for the trivial cases of the straight line (1) and the circle (4). Part of the reason for this is that such curves are rarities. Other than the trivial cases, there are also few sporadic (see [14] for a brief historic survey) higher order curves such as

$$\langle t^2, t^3 \rangle \qquad t \in [0, \infty] \tag{9}$$

giving rise to

$$\ell(t) = \frac{(9t^2+4)^{3/2} - 8}{27} \tag{10}$$

which can be readily inverted. The representation power provided by the few examples listed above is usually not sufficient.

With regard to the commonly used families, polynomials and splines, it is fairly easy to see by examining the leading coefficients, that there are no polynomial curves other than the straight line which satisfy condition (6). Further, as Farouki and Sakkalis [6] showed, this condition cannot even be satisfied by rational curves.

## 1.4 The Difficulties of Rectification and Inversion

How difficult is it to rectify and inverse a polynomial or a rational curve in order to obtain its arc length parameterization? The answer is not very encouraging. For polynomial curves of degree $n \geq 3$, the integral required for rectification (2) is not in general any elementary function. For $n = 3$ this is an elliptic integral [8], for $n = 2$ it is an elementary function describing the parabolic arc, and for $n = 1$ it is a linear function. General polynomial curves susceptible to rectification are only those of degree $n <= 2$. There are $2n - 3$ degrees of freedom in determining the shape of a polynomial curve of a degree $n$. For general quadratic curves, $n = 2$, we get a single degree of shape freedom. Regrettably, the arc length for the general quadratic curves is quite unweildy:

$$\begin{aligned}\ell(t) \;&=\; \left(\tfrac{b}{4a} + \tfrac{t}{2}\right)\sqrt{at^2+bt+c} \;+ \\ &\quad \tfrac{4ac-b^2}{8a^{3/2}} \ln\left(2at + b + 2\sqrt{a(at^2+bt+c)}\right)\end{aligned} \tag{11}$$

for some constants $a$, $b$ and $c$. The inversion of such a function is intractable by elementary methods.

The situation for general rational curves is no simpler since the function $\ell(t)$ is algebraic only in very demanding conditions. As a result of these difficulties, practitioners resorted to using numerical methods for computing the arc length [16, 9], and for equally spaced tracing of a curve [17, 18, 7].

Another approach of coping with the problematics of arc length parameterization is that taken by Farouki and Sakkalis [5] who coined the term *Pythagorean hodographs* for polynomial curves for which $\ell(t)$ is also a polynomial. A general method for the construction of such curves and an investigation of their fundamental properties also appears in [5]. A study of the applicability of Pythagorean hodographs for practical use in interpolation problems is given in [2]. It is possible to compute an explicit arc length parameterization. To do so, write the polynomial equation

$$s - \ell(t) = 0 \ , \tag{12}$$

and carry out the inversion by expressing its roots as a function of $s$.

Pythagorean hodographs of degree $n$ have $n-2$ degrees of shape freedom. For $n = 3$, Equation (12) is quadratic. By solving it, we get a class of arc length parameterized curves which has a single degree of shape freedom. Two degrees of shape freedom can be obtained by analytically solving the cubic equation that (12) becomes when $n = 4$. However, this solution is prohibitively complicated.

Quintic Pythagorean hodograph curves yielding three degrees of shape freedom were required in [2] for solving the interpolation problem considered there. For these curves, arc-length parameterization can still be obtained but only at the cost of using the mammoth explicit solution of the quartic equation! As there is no closed form solution for quintic and higher degree polynomial equations, it is impossible to generate richer classes of explicitly and elementarily arc-length parameterized curves using Pythagorean hodographs.

Yet another approach taken by Sakkalis and Farouki [14] was to find curves other than the Pythagorean hodographs for which $\ell(t)$ is an algebraic function. They were able to show, that whenever this happens, $\ell(t)$ is a square root of a polynomial. They also gave a full characterization of these *algebraically rectifiable polynomials*. Specifically, algebraically rectifiable cubics are essentially of the form $\langle t^2, kt^3 \rangle$ which is the generalization of (9). For these we have

$$\ell(t) = \frac{(9k^2t^2+4)^{3/2} - 8}{27k^2} \ , \tag{13}$$

and with the simple inversion

$$\ell^{-1}(t) = \frac{\sqrt{(27k^2t+8)^{2/3} - 4}}{3k} \ , \tag{14}$$

we get yet another class with one degree of shape freedom of arc length parameterized curves.

There are two classes of algebraically rectifiable quartics, both of which have two degrees of shape freedom. The first class has a relatively simple characterization by choice of parameters. However, its arc length function cannot be inverted as it is the square root of an eighth degree polynomial. The characterization of the second class of algebraically rectifiable quartics, as well as that of the algebraically rectifiable quintics and higher, is more complicated because a selection of a curve from these classes can only be carried out by solving a set of non-linear equations. Needless to say, the inversion of the arc length function for these classes is infeasible as well.

## 2 The New Family of Arc Length Parameterized Curves

The frustration in finding simultaneously simple and rich mathematical representation for a natural geometrical property, has lead us to initiate the research with the following objective: "Introduce a family of differentiable curves with an infinite number of degrees of freedom such that all curves in the family have a simple arc length parameterization." We have succeeded in constructing such a family and it is our hope that the entailing research will propound its exploitation in modeling and other interpolation applications.

As explained above, polynomials and rational curves proved fruitless in the context of the set objective. We have taken the approach of examining curves whose hodograph is rational, i.e.,

$$\mathbf{r}'(t) = \left\langle \frac{\alpha(t)}{\nu(t)}, \frac{\beta(t)}{\nu(t)} \right\rangle , \tag{15}$$

where $\alpha(t)$, $\beta(t)$ and $\nu(t)$ are polynomials. A necessary and sufficient condition for $\mathbf{r}(t)$ to be in its arc length parameterization form in this case is

$$\alpha^2(t) + \beta^2(t) = \nu^2(t) . \tag{16}$$

This condition is remarkably similar to the famous diophantic Pythagorean equation

$$a^2 + b^2 = c^2 . \tag{17}$$

Not surprisingly, it can be shown [12] that the general solution of the polynomial Pythagorean equation (16) parallels that of the diophantic Pythagorean equation (17)

$$\begin{array}{rcl} \alpha(t) & = & w_0(t)\left(w_1^2(t) - w_2^2(t)\right) \\ \beta(t) & = & 2w_0(t)w_1(t)w_2(t) \\ \nu(t) & = & w_0(t)\left(w_1^2(t) + w_2^2(t)\right) \end{array} , \tag{18}$$

where $w_0(t)$, $w_1(t)$ and $w_2(t)$ are arbitrary polynomials. For the purpose of (15), it is safe to assume that $w_0(t) \equiv 1$, that $w_1(t)$ and $w_2(t)$ are relatively prime, and that the leading coefficient of (say) $w_1$ is 1. Thus, we can write

$$\mathbf{r}'(t) = \left\langle \frac{w_1^2(t) - w_2^2(t)}{w_1^2(t) + w_2^2(t)}, \frac{2w_1(t)w_2(t)}{w_1^2(t) + w_2^2(t)} \right\rangle . \tag{19}$$

Conversely, the above form could have been reached by starting with the circle curve

$$\mathbf{r}(u) = \left\langle \frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2} \right\rangle \tag{20}$$

which satisfies $|\mathbf{r}(u) \equiv 1|$ for all $u$ and in particular also for $u = w_1(t)/w_2(t)$. By carrying out this substitution (21) can be derived. The approach based on the polynomial Pythagorean equation that we have taken shows that *all* unit speed rational hodograph curves can be obtained from (20) by such substitution.

Arc length parameterized curves with an arbitrary number of degrees of freedom can therefore be constructed by integration

$$\left\langle \int \frac{w_1^2(\tau) - w_2^2(\tau)}{w_1^2(\tau) + w_2^2(\tau)} d\tau, \int \frac{2w_1(\tau)w_2(\tau)}{w_1^2(\tau) + w_2^2(\tau)} d\tau \right\rangle . \tag{21}$$

The integrands are both rational functions and therefore, by a fundamental theorem of the integral calculus, given the roots of the polynomial $w_1^2(\tau) + w_2^2(\tau)$, a standard procedure yields an expression of the integrals as elementary functions.

For $w_1$ of degree $n_1$ and $w_2$ of degree $n_2$, there are $n_1 + n_2 + 3$ degrees of freedom in the construction (21): $n_1 + n_2 + 1$ for the polynomials coefficients, and two for the integration constants. Of these, only $n_1 + n_2 - 1$ are available for determining the shape of the curve. Two others are accounted for translation, one for rotation, and another one for the freedom in the parameterization corresponding to the substitution $t \to t + C$ which shifts the point from which the parameter is measured.

This construction can be generalized to higher dimensions. For example, for three-dimensional curves, we need the construct solutions for

$$\alpha^2(t) + \beta^2(t) + \gamma^2(t) = \nu^2(t) , \tag{22}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\nu(t)$ are polynomials. This can be done by the generalization of (18)

$$\begin{array}{rcl} \alpha(t) & = & w_0(t)\left(w_1^2(t) + w_2^2(t) - w_3^2(t)\right) \\ \beta(t) & = & 2w_0(t)w_1(t)w_3(t) \\ \gamma(t) & = & 2w_0(t)w_2(t)w_3(t) \\ \nu(t) & = & w_0(t)\left(w_1^2(t) + w_2^2(t) + w_3^2(t)\right) \end{array} . \tag{23}$$

With this, it is possible to write the three-dimensional equivalent of (19). By coordinate-wise integration, we get a construction of curves in arc-length parameterization form with an arbitrary degree of freedom. The extrapolation for even higher dimensions is straightforward.

We should not overlook the fact that the functions resulting from an integration of rational functions, as elementary as they may be, are still unwieldingly complicated. Fortunately, for the two-dimensional case, there is a more elegant construction of the same family of curves with a simpler characterization of the curves in it. Let us view the curves as residing in the complex plane with the $x$ and $y$ axes unified respectively with the real and the imaginary axes. A planar curve in this perspective is a complex function of a real variable. Rewriting (19) with it, we get

$$\begin{aligned} \mathrm{r}'(t) &= \left(\frac{w_1^2(t) - w_2^2(t)}{w_1^2(t) + w_2^2(t)} + i\frac{2w_1(t)w_2(t)}{w_1^2(t) + w_2^2(t)}\right) \\ &= \frac{w_1^2(t) + 2iw_1(t)w_2(t) - w_2^2(t)}{w_1^2(t) + w_2^2(t)} \\ &= \frac{(w_1(t) + iw_2(t))^2}{(w_1(t) + iw_2(t))\,(w_1(t) - iw_2(t))} \\ &= \frac{w_1(t) + iw_2(t)}{w_1(t) - iw_2(t)} \end{aligned} \tag{24}$$

For a complex polynomial $p(t)$, let $\overline{p}(t)$ denote its conjugate, i.e., the polynomial obtained from $p(t)$ by replacing all coefficients by their complex conjugate. Clearly, $\overline{w_1(t) + iw_2(t)} = w_1(t) - iw_2(t)$. We can therefore put (24) in a more concise form

$$\mathrm{r}'(t) = e^{i\theta}\frac{\overline{w}(t)}{w(t)} \ , \tag{25}$$

where $\theta$ is an arbitrary real constant and $w(t)$ is an arbitrary complex polynomial whose leading coefficient is 1. It will prove convenient to select $w(t)$ by its roots $\xi_1, \ldots, \xi_n$ rather than its coefficients. Observe that the roots of $\overline{w}(t)$ are $\overline{\xi}_1, \ldots, \overline{\xi}_n$.

We proceed by writing

$$\mathrm{r}'(t) = e^{i\theta}\left(1 + \frac{\overline{w}(t) - w(t)}{w(t)}\right) \ . \tag{26}$$

The degree of $\overline{w}(t) - w(t)$ is less than $n$. Therefore, Lagrange's fractional decomposition can be applied, to expand $\mathrm{r}'(t)$ about its poles, $\xi_1, \ldots, \xi_n$

$$\mathrm{r}'(t) = e^{i\theta}\left(1 + \sum_{1\le j\le n} \frac{R_j}{t - \xi_j}\right) \ , \tag{27}$$

where $R_j$ is the residue of $\mathrm{r}'(t)$ at the pole $\xi_j$

$$R_j = \frac{\overline{w}(\xi_i) - w(\xi_i)}{\prod\limits_{\substack{1\le k\le n \\ k\ne j}} (\xi_j - \xi_i)} = -\frac{2\Im(\xi_i)}{w'(\xi_i)} \tag{28}$$

(For simplicity of the presentation we assumed that $w(t)$ has no multiple roots.) With this formulation, the integration is rather straightforward

$$\mathrm{r}(t) = e^{i\theta}\sum_{j=1}^{n} R_j \ln(t - \xi_j) + C \ . \tag{29}$$

Recall that the complex function $\ln(\cdot)$ defined as

$$\ln re^{i\theta} = \ln(r) + i\theta$$

for a non-negative $r$ and real $\theta$, has multiple branches. Any of these branches could be legitimately selected for each of the terms $\ln(t - \xi_j)$ in (29). No confusion will arise since real $t$ guarantees that no crossing of a branch boundary will occur. It is also easy to check that branch selection corresponds to an additive constant. It is therefore sufficient to use the principal branch only.

Let us use the notation $\mathcal{G}_i$ for the class of curves obtained from $n$ degree polynomials with $n$ distinct roots. Counting again the degrees of freedom, we have $2n$ for $\xi_1, \ldots, \xi_n$, two for the integration constant $C$, and one for the rotation $\theta$ amounting to a total of $2n + 3$. Out of these, three are accounted for rigid motions: The integration constant $C$ corresponds to translation, and $\theta$ corresponds to rotation. Also, as before, one degree of freedom corresponds to a shift in the parameter, leaving only $2n - 1$ degrees of "shape freedom" in $\mathcal{G}_n$.

It should be understood that scaling is always counted as one of the degrees in shape freedom. (In our case, scaling can be simply carried out by multiplying all $\xi_j, \overline{\xi}_j$, $j = 1, \ldots n$ by a common factor.) Consequently, the class $\mathcal{G}_1$ has essentially only one curve which can be rotated, translated, and scaled as necessary for purpose of fitting.

# 3 General Arc Length Parameterized Curves

## 3.1 Degrees of Freedom

Suppose that $\mathrm{r}(t)$, $t \in [-\infty, \infty]$ is in arc-length parameterization form, i.e., satisfying the property $|\mathrm{r}'(t)| = 1$. This property is not dependent on the location of the origin nor on the orientation of the axes. Further, this property is preserved by uniform scaling of the curve, as described by $k\mathrm{r}(t/k)$ for some constant $k$. A single curve $\mathrm{r}(t)$ in arc length parameterization form gives rise to a family of arc-length parameterized curves which enjoys four degrees of freedom. When using a curve for an interpolation problem, two more degrees of freedom are available which are the selection of the beginning and end points of the interval. The class of curves spanned by $\mathrm{r}(t)$ can be written as

$$e^{i\theta}k\mathrm{r}(t/k) + \mathrm{c} \qquad [t_0, t_1] \ , \tag{30}$$

where c is an arbitrary vector and $\theta$, $k$, $t_0$ and $t_1$ are arbitrary scalars satisfying $-\pi \le \theta < \pi$, $k > 0$ and $t_0 < t_1$. (A rotation of $\pi$ about the origin is equivalent to scaling by a factor of $-1$. Hence there is no need to consider negative $k$'s.) If $\mathrm{r}(t)$ has no axes of symmetry then it also gives rise to an auxiliary class

$$e^{i\theta}k\overline{\mathrm{r}}(t/k) + \mathrm{c} \qquad [t_0, t_1] \ . \tag{31}$$

(The non-analyticity of the conjugate function is not a concern here since $\overline{\mathrm{r}}$ is nothing but a short hand for a pair of two real functions.) However, this will be of only marginal interest to us.

In counting degrees of *shape* freedom of a given class of curves, all of the six enumerated above are specifically excluded. It should be remarked that not even all these six are available for the two trivial examples of arc length parameterized curves. Circles are invariant under rotation—circle arcs therefore have just five degrees of freedom. Lines are invariant under scaling and translations in the direction of the line—line segments therefore have only four degrees of freedom. These are not sufficient for solving interesting interpolation problems.

Note however, that a single curve $\mathbf{r}(t)$ which is not invariant under translations, scaling, or general rotations, spans an interesting class of curves. Although this class has zero degrees of shape freedom, the six degrees it has may allow curve fitting in the setting of an interpolation problem with six constraints. Important interpolation problems of this type are those in which the two end points of the curve (four constraints) are given, together with two additional constraints: the slopes at the two end points, slope and curvature at one of the end points, or slope at one of the end points together with the length of the curve connecting them.

### 3.2 Slopes at Two End Points

Perhaps the most fundamental curve fitting problem is one in which we seek a curve segment which connects two points while assuming given derivatives at these points. A procedure for this can be used for a first order continuous interpolation of a sequence of points, where the slopes at each point in the sequence are given. To obtain such a procedure for the class with zero degrees of shape freedom spanned the by the arc length parameterized curve $\mathbf{r}(t)$, we must be able to solve the following set of equations

$$\begin{aligned} e^{i\theta} k\mathbf{r}(t_0/k) + \mathbf{c} &= \mathbf{p}_0 && (32)\\ e^{i\theta} k\mathbf{r}(t_0/k) + \mathbf{c} &= \mathbf{p}_1 && (33)\\ e^{i\theta}\mathbf{r}'(t_0/k) &= e^{i\theta_0} && (34)\\ e^{i\theta}\mathbf{r}'(t_1/k) &= e^{i\theta_1} && (35) \end{aligned}$$

for the unknowns $k$, $\theta$, $t_0$, $t_1$ and $\mathbf{c}$, where $\mathbf{p}_0$ and $\mathbf{p}_1$ are the two points and $\theta_0$ and $\theta_1$ are the slopes in them. By subtracting the first equation from the second we eliminate the unknown $\mathbf{c}$. By also applying the substitution $u_0 = t_0/k$ and $u_1 = t_1/k$ the set of equations takes the form

$$\begin{aligned} e^{i\theta} k\left(\mathbf{r}(u_1) - \mathbf{r}(u_0)\right) &= \mathbf{p}_1 - \mathbf{p}_0 && (36)\\ e^{i\theta}\mathbf{r}'(u_0) &= e^{i\theta_0} && (37)\\ e^{i\theta}\mathbf{r}'(u_1) &= e^{i\theta_1}\ . && (38) \end{aligned}$$

Only the first of the above equations is vectorial. By examining its magnitude part we can eliminate the unknown $k$,

$$k = \left|\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{r}(u_1) - \mathbf{r}(u_0)}\right| . \tag{39}$$

Since the selection of the direction of the $x$ axis is arbitrary, we may assume without loss of generality that $\arg(\mathbf{p}_1 - \mathbf{p}_0) = 0$. By dividing both sides of (36) by their absolute value we obtain:

$$e^{i\theta}\frac{(\mathbf{r}(u_1) - \mathbf{r}(u_0))}{|\mathbf{r}(u_1) - \mathbf{r}(u_0)|} = 1\ . \tag{40}$$

Dividing (37) and (38) by the above we get two scalar equations with unknowns $u_0$ and $u_1$:

$$\frac{\mathbf{r}'(u_j)|\mathbf{r}(u_1) - \mathbf{r}(u_0)|}{\mathbf{r}(u_1) - \mathbf{r}(u_0)} = e^{i\theta_j} \quad j = 0, 1 \tag{41}$$

Although these equations do not have in general an analytical solution, numerical solution is in many cases feasible. The right-hand side of the equations is nothing but the angle between the slopes at the end points and the line that connects these points. The left-hand side of the equations does not depend on the input constraints. We can therefore use these equations to span the range of boundary conditions in which $\mathbf{r}(t)$ yields a solution to the interpolation problem: This is done by tracing the pair $\langle\theta_0, \theta_1\rangle$ as $u_0$ and $u_1$ change in their respective domains. For example, one could fix $u_0$ and draw the parametric curve of $\langle\theta_0, \theta_1\rangle$ as $u_1$ changes, then increment $u_0$ by some $\epsilon$ and repeat the process. This trace yields a map between $\langle u_0, u_1\rangle$ and $\langle\theta_0, \theta_1\rangle$. If this map is stored, then it also gives a good initial approximation to be used for a numerical solution to the equations.

Note that if there is a closed form for the inverse of the function $\mathbf{r}'$, then equations (37) and (38) can be used to express $u_0$ and $u_1$ as a function of $\theta$:

$$\begin{aligned} u_0 &= \mathbf{r}'^{-1}\left(e^{i(\theta_0 - \theta)}\right) && (42)\\ u_1 &= \mathbf{r}'^{-1}\left(e^{i(\theta_1 - \theta)}\right)\ . && (43) \end{aligned}$$

Let $\hat{\mathbf{r}}$ be the complex function of real argument defined by $\mathbf{r}\circ\mathbf{r}'^{-1}$. By substituting the above two equations into (40) we get

$$e^{i\theta}\frac{\hat{\mathbf{r}}\left(e^{i(\theta_0-\theta)}\right) - \hat{\mathbf{r}}\left(e^{i(\theta_0-\theta)}\right)}{\left|\hat{\mathbf{r}}\left(e^{i(\theta_1-\theta)}\right) - \hat{\mathbf{r}}\left(e^{i(\theta_1-\theta)}\right)\right|} = 1\ , \tag{44}$$

thereby reducing the curve fitting problem to the task of solving a single (perhaps transcendental) equation with a single unknown. In contrast with the two equations and the two unknowns formulation presented above, this equation takes a different form for different values of the boundary conditions of the interpolation problem. Thus, no pre-processing can be used to compute initial approximation for the solution.

# 4 Interpolations using the class $\mathcal{G}_1$

In this section, we present the application of curves in $\mathcal{G}_1$ for solving interpolation problems. The setting used for the demonstration is that of two given end points and specified tangents at these points. The derivative vector of functions in the class $\mathcal{G}_1$ is in the form:

$$\mathbf{r}'(t) = e^{i\theta}\frac{t - \bar{\xi}}{t - \xi} \tag{45}$$

where $\xi$ is some complex number.

For any parameterization, if $t \in [-\infty, \infty]$, then the substitution $t \to t + C$ for any constant $C$, gives rise to the same point loci. We can therefore assume, without loss of generality, that $\Re(\xi) = 0$ and $\Im(\xi) = L$. Thus,

$$\mathbf{r}'(t) = e^{i\theta}\frac{t+iL}{t-iL} \; . \tag{46}$$

Integration yields

$$\mathbf{r}(t) = e^{i\theta}\left(t + 2iL\ln(t-iL)\right) + C \; . \tag{47}$$

In the interpolation problem we are given $\mathbf{p}_0$ and $\mathbf{p}_1$, points in the plane, and tangents' slopes $\theta_0$ and $\theta_1$. We are to find $t_0$, $t_1$, $L$, $\theta$ and $C$ such that the following boundary conditions are satisfied:

$$\mathbf{r}(t_0) = \mathbf{p}_0 \tag{48}$$

$$\mathbf{r}(t_1) = \mathbf{p}_1 \tag{49}$$

$$\mathbf{r}'(t_0) = e^{i\theta_0} \tag{50}$$

$$\mathbf{r}'(t_1) = e^{i\theta_1} \; . \tag{51}$$

Substituting (46) into (50) and (51) yields

$$e^{i\theta}\frac{t_0+iL}{t_0-iL} = e^{i\theta_0} \tag{52}$$

$$e^{i\theta}\frac{t_1+iL}{t_1-iL} = e^{i\theta_1} \; . \tag{53}$$

The effects of scaling are tacitly accounted for by using a canonical reparametrization

$$u = t/L \; . \tag{54}$$

With this parameterization,

$$e^{i\theta}\frac{u_0+i}{u_0-i} = e^{i\theta_0} \tag{55}$$

$$e^{i\theta}\frac{u_1+i}{u_1-i} = e^{i\theta_1} \; , \tag{56}$$

or in angular form

$$\begin{aligned} \theta + 2\arg(u_0+i) &= \theta_0 \\ \theta + 2\arg(u_1+i) &= \theta_1 \end{aligned} \tag{57}$$

By subtracting (48) from (49) and using the canonical parameterization we have

$$e^{i\theta}L\left(u_1 - u_0 + 2i\ln\frac{u_1-i}{u_0-i}\right) = p_1 - p_0 \; . \tag{58}$$

The above is an equation of complex values. Therefore, both the absolute value and the angular part of both sides must be equal. We can therefore ignore the absolute value now. Later, when the value of $u_0$, $u_1$ and $\theta$ is determined, the absolute value equation can be used to easily determine the value of $L$. The angular form of (58) is

$$\theta + \arg(u_1 - u_0 + 2i\ln(\frac{u_1-i}{u_0-i})) = 0 \; . \tag{59}$$

Now we can use (57) to express $u_0$ and $u_1$ in terms of $\theta$. Doing so and substituting into the above will give us a one-variable transcendental equation for $\theta$

$$\theta + \arctan\left(\frac{2\ln\sin\frac{\theta-\theta_0}{2} - 2\ln\sin\frac{\theta-\theta_1}{2}}{\theta_1 - \theta_0 + \cot\frac{\theta-\theta_0}{2} - \cot\frac{\theta-\theta_1}{2}}\right) = 0 \; . \tag{60}$$

This equation can be easily solved numerically for specific values of $\theta_0$ and $\theta_0$. (Some results we have, which are excluded from this research proposal, give indication as to when the solution exists and when it is multiple.) It is important to understand that once such an interpolating curve is found, admittedly with some numerical iterative effort, all subsequent manipulation and rendering of the curve can be done analytically. This is in contrast to cubics, in which the curve could be determined analytically, but then, *each* rendering step requires a numerical iterative process.

## 4.1 Examples

We have numerically implemented an algorithm which, given two slopes, constructs an arclength parametrized curve with these slopes. Without loss of generality, we assume that the curve starts at $(0,0)$ and ends at $(1,0)$. The curve belongs to the aforedescribed class $\mathcal{G}_1$; hence, using the previous notation, $\mathbf{p}_0 = (0,0)$, $\mathbf{p}_1 = (1,0)$, the input consists of $\theta_0, \theta_1$, and the values of $u_o, u_1, k, \theta, C$ are sought. In Figures 1 and 2, two examples are provided.

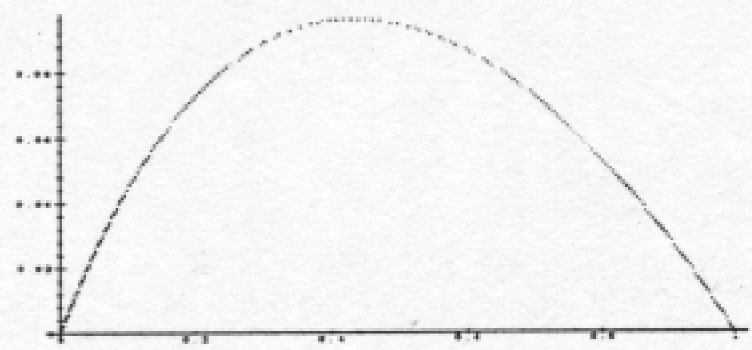

Figure 1: $\mathcal{G}_1$ curve with $\theta_0 = 0.5$, $\theta_1 = -0.3$. The solution is $u_0 = 1.15, u_1 = 3.08, k = 0.53, \theta = -0.92, C = -0.18 - 0.50I$.

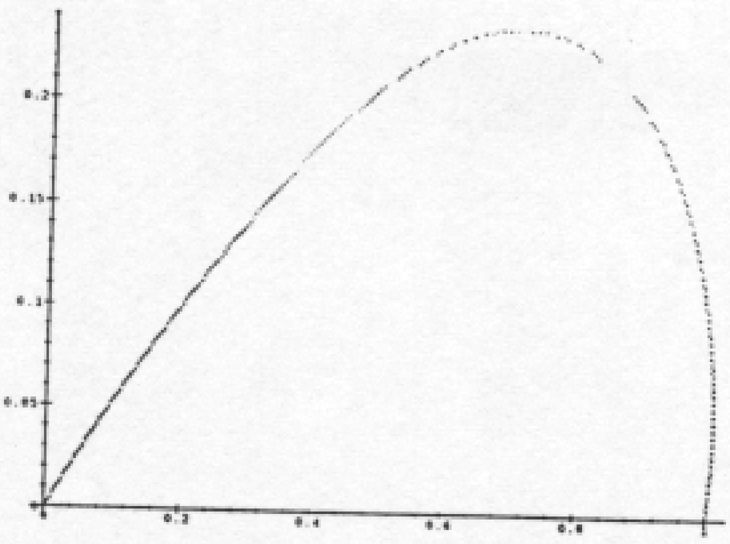

Figure 2: $\mathcal{G}_1$ curve with $\theta_0 = 0.5, \theta_1 = -2.0$. The solution is $u_0 = -5.79, u_1 = -0.15, k = 0.21, \theta = 0.84, C = 0.98 - 0.02I$.

## 5 Conclusions and Further Research

A novel family of curves was described, which allows to construct arc length parameterized curves between two arbitrary points, with given slopes at the curves' two endpoints. A numerical routine for finding the curves was developed and implemented. Future work may include solving more general problems, such as extending the concepts introduced here to three-dimensional curves.

## References


[1] B. A. Barsky and T. D. Rose. Parametric curves tutorial: Geometric continuity of parametric curves: Three equivalent characterizations. *IEEE Computer Graphics & Applications*, 9(11):60–68, Nov. 1989.

[2] R. T. Farouki. Pythagorean-hodograph curves in practical use. In R. E. Bamhill, editor, *Geometric Processing for Design and Manufacturing*. SIAM, 1992.

[3] R. T. Farouki and C. A. Neff. Algebraic properties of plane offset curves. *Computer Aided Geometric Design*, 7:101–127, 1990.

[4] R. T. Farouki and C. A. Neff. Analytic properties of plane offset curves. *Computer Aided Geometric Design*, 7:83–99, 1990.

[5] R. T. Farouki and T. Sakkalis. Pythagorean hodographs. *IBM J. Res. Dev.*. 34(5):736–751, Sept. 1990.

[6] R. T. Farouki and T. Sakkalis. Real rational curves are not 'unit speed'. *Computer Aided Geometric Design*, 8:151–157, 1991.

[7] Fosters. Curvature-dependent parametrization of curves and surfaces. *Computer Aided Design*, 23(8):569–578, Oct. 1991.

[8] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, Inc., 1250 Sixth Avenue, San Diego, California 92101, USA, corrected and enlarged edition (prepared by alan jeffrey edition, 1979.

[9] B. Guenter and R. Parent. Computing the arc length of parametric curves. *IEEE Computer Graphics & Applications*, 10(5):72–78, May 1990.

[10] J. Hoschek. Offset curves in the plane. *Computer Aided Design*, 17(2):77–82, Mar. 1985.

[11] E. Kreyszig. *Differential Geometry*. University of Toronto Press, Toronto, Canda, 1959.

[12] K. K. Kubota. Phythagorean triples in unique factorization domains. *Americ. Math. Monthly*, 5:503–505, 1972.

[13] J. D. Lawrence. *A Catalog of Special Plane Curves*. Dover Publications, Inc., 180 Varick Street, New York, NY 10014, USA, 1972.

[14] T. Sakkalis and R. T. Farouki. Algebraically recitifiable parametric curves. Technical Report RC17973, IBM Thomas J. Watson Research Center, P. O. Box 218, Yorktown Heights, NY 10598, 1992.

[15] T. W. Sederberg and X. Wang. Rational hodographs. *Computer Aided Geometric Design*, 4:333–335, 1987.

[16] R. J. Sharpe and R. W. Thorpe. Numerical method for extracting and arc length parameterization from parametric curves. *Computer Aided Design*, 12(2), Mar. 1982.

[17] C. Wayne Mastin. Parametrization in grid generation. *Computer Aided Design*, 18(1):22–24, Jan./Feb. 1986.

[18] U. Wever. Optimal parameterization for cubic splines. *Computer Aided Design*, 23(9):641–644, Nov. 1991.